Discussion Papers of the
Max Planck Institute for
Research on Collective Goods
2021/19

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October 2021

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Public-Good Provision with Macro Uncertainty about Preferences: Efficiency, Budget Balance, and Robustness*

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October 11, 2021

Abstract

The paper studies efficient public-good provision in a model with private values whose distribution depends on a macro shock; conditionally on this shock, values are independent and identically distributed. A generalization of the Bayesian mechanism of d'Aspremont and Gérard-Varet is shown to implement an efficient provision rule with budget balance. However, first-best implementation and budget balance are incompatible with a reqruirement of weak robustness whereby incentive compatibility of the mechanism is independent of the stochastic specification within the class of specifications defined by the structure of the model. Budget imbalances with robust implementation are small if there are many participants, as surplus from the Clarke-Groves mechanism converges to zero in probability when the number of participants becomes large. In the limit, with a continuum of agents, a first-best provision rule with equal cost sharing is robustly incentive-compatible. In this limit, information about the macro shock, which is the only thing that matters for public-good provision, can be elicited without any efficiency loss.

Key Words: Efficient public-good provision, incomplete information, conditionally independent private values, macro uncertainty, budget balance, weakly robust incentive compatibility.

JEL: D60, D82, H41.

*I am grateful to Felix Bierbrauer and Tilman Börgers for helpful discussions and comments.

1 Introduction

This paper considers the scope for efficient public-good provision with incomplete information when the participants' preferences depend on an unobserved macro shock. Whereas in standard Bayesian models of public-good provision under incomplete information the preference parameters of the different participants are given by independent random variables, I assume that preference parameters are *conditionally* independent and identically distributed given some random variable $\tilde{y}$, which represents a macro shock.

Macro uncertainty adds an important dimension to the information problem in public-good provision. For suppose that a public good comes as a single indivisible unit, say at a cost K_n when there are n participants, and let $\theta_1, ..., \theta_n$ be the value of the public good to participants $1, ..., n$. An efficient provision rule requires the public good to be provided if $\sum_{i=1}^{n} \theta_i > K_n$ and not to be provided if $\sum_{i=1}^{n} \theta_i < K_n$. Thus, if K_n takes the form $K_n = n \cdot k$, the question is whether $\frac{1}{n} \sum_{i=1}^{n} \theta_i$ is greater or less than k. If the preference parameters $\theta_1, ..., \theta_n$ are the realizations of independent and identically distributed random variables $\tilde{\theta}_1, ..., \tilde{\theta}_n$, then, with a probability close to one, the average $\frac{1}{n} \sum_{i=1}^{n} \theta_i$ will be close to the expected value $E\tilde{\theta}_i$ if n is large. In this case, if $E\tilde{\theta}_i > k$, the expected efficiency loss from a non-contingent decision to provide the public good will be small; similarly, if $E\tilde{\theta}_i < k$, the expected efficiency loss from a non-contingent decision not to provide the public good will be small.

In contrast, with macro uncertainty, the expected efficiency loss from such non-contingent decisions on public-good provision are non-negligible even if n is large. If $\theta_1, ..., \theta_n$ are the realizations of random variables $\tilde{\theta}_1, ..., \tilde{\theta}_n$ that are conditionally independent and identically distributed given $\tilde{y}$, then, with a probability close to one, the average $\frac{1}{n} \sum_{i=1}^{n} \theta_i$ will be close to the *conditional* expectation $E[\tilde{\theta}_1|\tilde{y}]$ so an efficient provision decision hinges on whether $E[\tilde{\theta}_1|\tilde{y}]$ is greater or smaller than k. This question is nontrival even if n is large.

For models with conditionally independent and identically distributed characteristics, the paper provides the following results. First, a straightforward generalization of the incentive mechanism of d'Aspremont and Gérard-Varet (1979) is shown to provide for efficient public-good provision with *ex post* budget balance. Like the mechanism of d'Aspremont and Gérard-Varet, this mechanism satisfies *ex ante* but not *interim* participation constraints. If *interim* participation constraints were imposed, budget balance would be lost.[1]

Even without *interim* participation constraints, the incentive mechanisms that provide for efficient public-good provision with *ex post* budget balance are not robust to changes in the stochastic specification. A further result of the paper shows that efficient public-good provision and *ex post* budget balance are incompatible if incentive mechanisms must satisfy a weak robustness condition requiring incentive compatible to be preserved when the probability distribution of the aggregate shock or the mapping from aggregate shocks to conditional

[1] This conclusion follows by a straightforward extension of the argument given in Güth and Hellwig (1986) for the case of independent private values.

distributions of characteristics are changed.

However, budget imbalances from robust implementation of an efficient provision rule can be small if the number n of participants is large.[2] With a large population, individual incentive compatibility conditions pose no problem for the implementation of an efficient provision rule through a robustly incentive-compatible direct mechanism with budget balance.[3]

The assumption that participants' characteristics are not only conditionally independent but also conditionally identically distributed is imposed for simplicity. Without this additional assumption, the results of this paper would still go through, but the notation and the analysis itself would be more complicated.

However, the assumption of conditionally independent and identically distributed preference parameters is of interest in iteself because it is equivalent to the assumption that the preference parameters $\tilde{\theta}_1, ..., \tilde{\theta}_n$ belong to an infinite sequence of *exchangeable* random variables, whose joint distribution is unchanged under any permutation of their labels.[4] Exchangeability reflects a notion of anonymity, whereby players' names are irrelevant for the relations between them.[5]

In the following, Section 2 presents the model, Section 3 the results on Bayesian implementation. Section 4 introduces the concept of weak robustness and formulates the impossibility theorem stating that no mechanism can provide for weakly robust implementation of efficient public-good provision with budget balance. Section 5 discusses the behaviour of budget imbalances with robust implementation when the number of participants is large (though finite). Section 6 sketches the arguments for the limit economy with a continuum of participants. All proofs are given in the Appendix.

2 Bayesian Implementation

Consider a model with $n > 1$ agents, one private good and one public good that comes as a single indivisible unit. Assume that the installation of the public good costs K units of the private good. If the public good is installed, all agents enjoy it; there is no scope for individual exclusion and no problem of crowding. Agent i obtains the net payoff

$$\theta_i \cdot q - t_i, \tag{2.1}$$

[2]This result extends and strengthens the analysis of Green and Laffont (1979). Focussing on Clarke-Groves mechanisms, i.e. robust mechanisms that never run deficits, they show that, as n goes out of bounds, the expected value of the budget surplus *per capita* goes to zero. For such mechanisms, I show that the budget surplus itself goes to zero in probability.

[3]In contrast, coalition incentive compatibility may preclude first-best implementation even if the population is large. In a model with single-peaked preferences on a linearly ordered space of outcomes, Hellwig (2021) shows that group strategy proofness, i.e., the dominant-strategy version of coalition incentive compatibility, is obtained if and only if the mechanism is equivalent to a combination of binary votes over neighbouring outcomes.

[4]See Diakonis and Freedman (1980), Hammond and Sun (2008).

[5]See Hellwig (forthcoming).

where θ_i is a preference parameter for agent i, $q \in \{0, 1\}$ is an indicator variable showing whether the public good is installed or not, and t_i is the agent's payment in units of the private good.

The preference parameter θ_i is the realization of a random variable $\tilde{\theta}_i$ that takes values in an interval $[0, \bar{\theta}]$. The preference parameters $\tilde{\theta}_1, ..., \tilde{\theta}_n$ are assumed to be conditionally independent and identically distributed given some random variable $\tilde{y}$ that takes values in some separable metric space Y. I write $F(\cdot|\tilde{y})$ for the (regular) conditional distribution of $\tilde{\theta}_i$ given $\tilde{y}$ and G for the probability distribution of $\tilde{y}$.[6] The marginal probability distribution of $\tilde{\theta}_i$,

$$F(\cdot) = \int_Y F(\cdot|y)\, dG(y), \tag{2.2}$$

is assumed to have full support $[0, \bar{\theta}]$.

The stochastic structure of the model, including the conditional distributions $F(\cdot|y), y \in Y$, and the probability distribution G, are assumed to be common knowledge. In addition, each agent i is informed about the realization of his own preference parameter $\tilde{\theta}_i$. The agent receives no additional information about the other agents.

The allocation problem is to determine under which conditions the public good is to be provided and under which conditions it is not to be provided. To ensure that this problem is non trivial, I assume that

$$0 < K < n\bar{\theta} \tag{2.3}$$

so that there is some positive probability that benefits from the public good are below per-capita costs for all agents and some positive probability that benefits from the public good are above per-capita costs for all agents.

To take account of such information, the provision decision must depend on the preference parameters of the participants. Since information about these parameters is private, one must rely on incentive mechanisms to elicit it in an incentive-compatible manner.

I consider incentive-compatible direct mechanisms. A *direct mechanism* is given by functions $q, t_1, ..., t_n$ with the interpretation that participants are asked the values of their preference parameters and, for any vector $(\hat{\theta}_1, ..., \hat{\theta}_n)$ of announced preference parameters, $q(\hat{\theta}_1, ..., \hat{\theta}_n) \in \{0, 1\}$ is the level of public-good provision and $t_i(\hat{\theta}_1, ..., \hat{\theta}_n)$ is the payment that the mechanism stipulates for agent i when the announcement vector is $(\hat{\theta}_1, ..., \hat{\theta}_n)$.

A direct mechanism is *Bayesian incentive-compatible* if, for each agent i, truthtelling is a Bayes-Nash best response to truthtelling by the other agents. Thus, for any θ_i, the report $\hat{\theta}_i = \theta_i$ must maximize the agent's net expected payoff

$$\theta_i \cdot Q(\hat{\theta}_i, \theta_i) - T_i(\hat{\theta}_i, \theta_i), \tag{2.4}$$

where

$$Q(\hat{\theta}_i, \theta_i) := \int_Y \int_{[0,\bar{\theta}]} q(\hat{\theta}_i, \boldsymbol{\theta}_{-i})\, dF^{n-1}(\boldsymbol{\theta}_{-i}|y)\, dG(y|\theta_i) \tag{2.5}$$

[6] An example would be the specification $Y = \mathcal{M}([0, \bar{\theta}])$ with $F(\cdot|\tilde{y}) = \tilde{y}$.

and

$$T_i(\hat{\theta}_i, \theta_i) := \int_Y \int_{[0,\bar{\theta}]} t_i(\hat{\theta}_i, \boldsymbol{\theta}_{-i})\; dF^{n-1}(\boldsymbol{\theta}_{-i}|y)\; dG(y|\theta_i) \tag{2.6}$$

are the agent's conditional expectations of $q(\hat{\theta}_i, \tilde{\boldsymbol{\theta}}_{-i})$ and $t_i(\hat{\theta}_i, \tilde{\boldsymbol{\theta}}_{-i}))$ given that $\tilde{\theta}_i = \theta_i$; in these expressions $F^{n-1}(\cdot|y)$ denotes the $n-1$-fold product of $F(\cdot|y)$ and $G(\cdot|\theta_i)$ denotes a conditional distribution for $\tilde{y}$ given that $\tilde{\theta}_i = \theta_i$.

If G is a degenerate distribution that assigns all mass to some point $\hat{y} \in Y$, one has $F(\cdot|\tilde{y}) = F(\cdot)$ almost surely, and the random variables $\tilde{\theta}_1, ..., \tilde{\theta}_n$ are actually independent. For this case, the following result is well known.[7]

Proposition 2.1 *Fix $\hat{y} \in Y$ and assume that G assigns all probability mass to the singleton $\{\hat{y}\}$. If $(q, t_1, ..., t_n)$ is a Bayesian incentive-compatible direct mechanism, then the associated functions $Q_1, ..., Q_n$ and $T_1, ..., T_n$ are independent of $\theta_1, ..., \theta_n$ and satisfy*

$$T_i(\hat{\theta}_i) = T_i(0) + \theta_i \cdot Q(\hat{\theta}_i) - \int_0^{\hat{\theta}_i} Q(\theta_i^{'})\; d\theta_i' \tag{2.7}$$

for all i and $\hat{\theta}_i$, where

$$Q_i(\hat{\theta}_i) := \int_{[0,\bar{\theta}]} q(\hat{\theta}_i, \boldsymbol{\theta}_{-i})\; dF^{n-1}(\boldsymbol{\theta}_{-i}|\hat{y}), \tag{2.8}$$

$$T_i(\hat{\theta}_i) := \int_{[0,\bar{\theta}]} t_i(\hat{\theta}_i, \boldsymbol{\theta}_{-i})\; dF^{n-1}(\boldsymbol{\theta}_{-i}|\hat{y}), \tag{2.9}$$

and the function $Q(\cdot)$ is nondecreasing. Conversely, if q is any nondecreasing function on $[0, \bar{\theta}]^n$, a Bayesian incentive-compatible direct mechanism $(q, t_1, ..., t_n)(\hat{y})$ is obtained by setting

$$\begin{aligned} t_i(\theta_1, ..., \theta_n|\hat{y}) = T_i(\theta_i|\hat{y}) + \frac{1}{n}\, K \cdot q(\theta_1, ..., \theta_n) - \frac{1}{n} \int K \cdot q(\theta_i, \boldsymbol{\theta}'_{-i})\; dF^{n-1}(\boldsymbol{\theta}'_{-i}|\hat{y}) \\ - \frac{1}{n-1} \sum_{j \neq i} \left[T_j(\theta_j|\hat{y}) - \int T_j(\theta_j'|\hat{y})\; dF(\theta_j'|\hat{y}) \right] \\ + \frac{1}{n-1} \sum_{j \neq i} \frac{1}{n} \left[\int K \cdot q(\theta_j, \boldsymbol{\theta}'_{-j})\; dF^{n-1}(\boldsymbol{\theta}'_{-j}|\hat{y}) - \int K \cdot q(\hat{\boldsymbol{\theta}})\; dF^n(\hat{\boldsymbol{\theta}}|\hat{y}) \right], \end{aligned} \tag{2.10}$$

for all i and all $(\theta_1, ..., \theta_n)$, where

$$T_i(\hat{\theta}_i|\hat{y}) = T_i(0|\hat{y}) + \theta_i \cdot Q_i(\hat{\theta}_i|\hat{y}) - \int_0^{\hat{\theta}_i} Q(\theta_i^{'}|\hat{y})\; d\theta_i' \tag{2.11}$$

and

$$Q_i(\hat{\theta}_i|\hat{y}) := \int_{[0,\bar{\theta}]^{n-1}} q(\hat{\theta}_i, \boldsymbol{\theta}'_{-i})\; dF^{n-1}(\boldsymbol{\theta}'_{-i}|\hat{y}) \tag{2.12}$$

for all i and all $\hat{\theta}_i$.

[7] See, e.g., Proposition 3.3 of Güth and Hellwig (1986).

For the general case of *conditionally* independent and identically distributed private values, i.e., for arbitrary G, the following result provides a straightforward extension of the second part of Proposition 2.1.

Proposition 2.2 *Let $F(\cdot|y), y \in Y$, and G be any distribution on Y. If q is any nondecreasing function on $[0, \bar{\theta}]^n$, a Bayesian incentive-compatible direct mechanism $(q, t_1, ..., t_n)$ is obtained by setting*

$$t_i(\theta_1, ..., \theta_n) = \int_Y t_i(\theta_1, ..., \theta_n|\hat{y})\, dG(\hat{y}) \tag{2.13}$$

for all i and all $(\theta_1, ..., \theta_n)$, where $t_i(\theta_1, ..., \theta_n|\hat{y})$ is given by (2.10). The associated functions Q_i and T_i satisfy

$$Q_i(\hat{\theta}_i, \theta_i) = \int_Y Q(\hat{\theta}_i|\hat{y})\, dG(\hat{y}|\theta_i) \tag{2.14}$$

and

$$T_i(\hat{\theta}_i, \theta_i) = \int_Y T(\hat{\theta}_i|\hat{y})\, dG(\hat{y}|\theta_i) \tag{2.15}$$

for all $i, \hat{\theta}_i$, and θ_i, where $Q_i(\hat{\theta}_i|\hat{y})$ and $T_i(\hat{\theta}_i|\hat{y})$ are given by (2.12) and (2.11).

Proposition 2.2 generalizes the second part of Proposition 2.1 to allow for aggregate uncertainty, represented by the random variable $\tilde{y}$. For the first part of Proposition 2.1, such a generalization is not available because some Bayesian incentive-compatible mechanisms do not admit integral representations along the lines of (2.7); for examples, see Crémer and McLean (1988), McAfee and Reny (1992), or Kosenok and Severinov (2008).

3 First-Best Implementation

For a result on efficient implementation, Proposition 2.2 is all that is needed. A public-good provision rule q is said to be *first-best* if, for each vector $(\theta_1, ..., \theta_n) \in [0, \bar{\theta}]^n$ of preference parameters, the provision level $q(\theta_1, ..., \theta_n)$ maximizes the aggregate surplus

$$\sum_{i=1}^{n} \theta_i \cdot q - K \cdot q. \tag{3.1}$$

Under a first-best provision rule, the provision level $q(\theta_1, ..., \theta_n)$ is obviously nondecreasing in $\theta_1, ..., \theta_n$. Because $q(\cdot)$ is nondecreasing, Proposition 2.2 characterizes a Bayesian incentive-compatible direct mechanism that implements $q(\cdot)$.

The following result uses this characterization to generalize the finding of d'Aspremont and Gérard-Varet (1979) that Bayesian implementation of a first-best public-good provision rule is compatible with budget balance. A mechanism satisfies *budget balance*, if the payment functions $t_1, ..., t_n$ are specified so that the balanced-budget condition

$$\sum_{i=1}^{n} t_i(\theta_1, ..., \theta_n) = K \cdot q(\theta_1, ..., \theta_n) \tag{3.2}$$

holds for all $(\theta_1, ..., \theta_n) \in [0, \bar{\theta}]^n$.

Proposition 3.1 *Let G be any distribution on Y. If q is a first-best public-good provision rule, there exists a Bayesian incentive-compatible direct mechanism that implements q and satisfies budget balance.*

Because of budget balance, the sum of *ex ante* net expected payoffs of the different participants satisfies

$$\begin{aligned} &\sum_{i=1}^{n} \int [\theta_i \cdot q(\theta_1, ..., \theta_n) - t_i(\theta_1, ..., \theta_n)] \, dF^n(\boldsymbol{\theta}|\hat{y}) \, dG(\hat{y}) \\ &= \int \left[\sum_{i=1}^{n} \theta_i - K\right] \cdot q(\theta_1, ..., \theta_n) \, dF^n(\boldsymbol{\theta}|\hat{y}) \, dG(\hat{y}). \end{aligned} \tag{3.3}$$

This expression is strictly positive because, for a first-best provision rule q, the integrand is nonnegative with probability one and positive with positive probability. With a suitable symmetry property of the contribution functions $t_1, ..., t_n$, the *ex ante* net expected payoffs of the different participants must all be the same and must have the same sign as the sum of *ex ante* net expected payoffs over all participants. In this case, acceptance of the incentive mechanism $(q, t_1, ..., t_n)$ in Proposition 3.1 is *ex ante* individually rational.

However, acceptance of the incentive mechanism $(q, t_1, ..., t_n)$ in Proposition 3.1 is not *interim* individually rational for all agents. This result follows from a straightforward extension of Proposition 5.4 in Güth and Hellwig (1986).[8]

First-best implementation with *interim* individual rationality can be achieved if budget balance is not required. For suppose that, instead of (2.10), the payment functions $t_1, ..., t_n$ satisfy (2.13), as well as

$$t_i(\theta_i, \boldsymbol{\theta}_{-i}|\hat{y}) = T_i(\theta_i|\hat{y})$$

and

$$T_i(0|\hat{y}) = 0$$

for all i, θ_i, $\boldsymbol{\theta}_{-i}$, and $\hat{y}$. Suppose also that $T_i(\theta_i|\hat{y})$ is given by (2.11), with $Q_i(\cdot|\hat{y})$ given by (2.12) and the first-best provision rule q. The resulting mechanism is easily seen to implement q. Moreover, the *interim* expected net payoff (2.4)

[8]Similar results are also given by Rob (1988) and Mailath and Postlewaite (1990). All these results can be interpreted as variations of the impossibility theorem of Myerson and Satterthwaite (1983).

of any agent i with preference parameter θ_i under this mechanism is at least equal to $\theta_i \cdot Q(0, \theta_i) - T_i(0, \theta_i)$, the expected net payoff from reporting $\hat{\theta}_i = 0$, which is nonnegative because $\theta_i \cdot Q(0, \theta_i) \geq 0$ and $T_i(0, \theta_i) = 0$. However, this mechanism violates budget balance; it even violates the *ex ante* budget balance condition

$$\int \left[\sum_{i=1}^{n} t_i(\theta_1, ..., \theta_n) - K \cdot q(\theta_1, ..., \theta_n) \right] \, dF^n(\boldsymbol{\theta}|\hat{y}) \, dG(\hat{y}) = 0.$$

Interim individual rationality and incentive compatibility require some of the benefits of public–good provision to agents with high public-good valuations to be left with the participants so that these high valuations cannot fully be relied on for contributions to costs. As in other contexts, the information rents that are thus induced create a conflict between efficiency and budget balance. Such an arrangement is only possible if an outside party is willing to cover the deficit.

With correlated characteristics of the different particpants, for many specifications of the model,[9] one can use the devices of Crémer and McLean (1988), or McAfee and Reny (1992), to provide for efficient public-good provision with *interim* participation constraints and *ex ante* budget balance. Whether such mechanisms can also be designed to satisfy *ex post* budget balance, i.e. condition (3.2), is an open question.[10] In the following, I do not impose individual rationality constraints.

4 An Impossibility Theorem for Weakly Robust Implementation

The Bayesian approach to implementation has been criticized because it relies on information about the beliefs that agents form about the other agents' preference parameters given their own preference parameters. The assumption that this information is available to the mechanism designer is highly implausible. Several authors have therefore proposed robustness requirements for incentive mechanisms.[11]

A direct mechanism $(q, t_1, ..., t_n)$ is said to be *robustly incentive-compatible* if it is Bayesian incentive-compatible for all priors Φ on $[0, \bar{\theta}]^n$. Incentive com-

[9] The word "many" refers to the findings of Chen and Xiong (2013) and Gizatulina and Hellwig (2017) that the set of specifications allowing the use of Crémer-McLean mechanisms for preference elicitation without information rents is generic ("large") in a topological sense. The question of genericity must be distinguished from the question of robustness that is considered in the next section. In fact, the Crémer-McLean mechanisms for preference elicitation tend to be highly nonrobust.

[10] One easily verifies that, because of the special structure with conditionally independent and identically distributed characteristics, the sufficient condition that Kosenok and Severinov (2008) give for first-best implementation with *ex post* budget balance is not satisfied here. Whereas Kosenok and Severinov (2008) refer to this condition as being necessary as well as sufficient, the necessity result comes with a quantifier requiring balanced-budget implementability for *all* utility specifications.

[11] See Ledyard (1978), Bergemann and Morris (2005), Börgers and Smith (2014).

patibility of such a mechanism is independent of the conditional distributions that determine agents' beliefs about the characteristics of other agents.

In the present context, robust incentive compatibility is too strong a requirement. Many priors Φ on $[0, \bar{\theta}]^n$ do not have conditionally independent and identically distributed characteristics of agents. As mentioned in the introduction, this property is implied by the assumption that $\tilde{\theta}_1, ..., \tilde{\theta}_n$ belong to an infinite sequence of exchangeable random variables. If this property is taken to be commonly known, there is no point in asking for Bayesian incentive compatibility at priors that fail this assumption.

I therefore use a weaker concept of robustness. A direct mechanism $(q, t_1, ..., t_n)$ is *weakly robustly incentive-compatible* if it is Bayesian incentive-compatible for all priors Φ on $[0, \bar{\theta}]^n$ such that, for some separable metric space Y, some distribution G_Φ on Y, and some measurable mapping $y \longmapsto F_\Phi(\cdot|y)$ from Y to the space of probability measures on $[0, \bar{\theta}]$,

$$\Phi(B) = \int_Y (F_\Phi)^n(B|y)dG_\Phi(y),$$

for all measurable sets $B \subset [0, \bar{\theta}]^n$, where $(F_\Phi)^n(\cdot|y)$ is the n-fold product of the measure $F_\Phi(\cdot|y)$. Weak robustness implies that the incentive compatibility of a direct mechanism must not depend on the probability distribution G_Φ on Y or on the mapping $y \longmapsto F_\Phi(\cdot|y)$ from Y to the space of probability measures on $[0, \bar{\theta}]$.

Lemma 4.1 *A direct mechanism $(q, t_1, ..., t_n)$ is weakly robustly incentive-compatible if and only if it is Bayesian incentive-compatible for all priors Φ on $[0, \bar{\theta}]^n$ that take the form $\Phi = (F_\Phi)^n$ where $(F_\Phi)^n$ is the n-fold product of the measure F_Φ on $[0, \bar{\theta}]$.*

As is well known, robust implementability is equivalent to implementability in dominant strategies, i.e., a mechanism $(q, t_1, ..., t_n)$ is robustly incentive-compatible if and only if truthtelling is a dominant strategy.[12] Since weak robustness is obviously implied by robustness, it follows that any mechanism that is implementable in dominant strategies is also weakly robustly incentive-compatible. I do not know whether the converse is also true, i.e. whether weakly robust incentive compatibility also implies implementability in dominant strategies.

However, with an additional condition of *anonymity*, weakly robust implementability of first-best public-good provision rules is incompatible with budget balance. A direct mechanism $(q, t_1, ..., t_n)$ is said to be *anonymous* if $q(\theta_1, ..., \theta_n)$ depends only on the cross-section distribution of the preference parameters and, for any i, $t_i(\theta_1, ..., \theta_n)$ depends only on the preference paramter θ_i of agent i and on the distribution of the other agents' preference parameters and, moreover, the dependence takes the same form for all i.

[12] See Bergemann and Morris (2005). An application to public-provision is provided in Bierbrauer and Hellwig (2016). Börgers and Smith (2014) argue that implementability in dominant strategies is too strong a requirement and propose a weaker concept of undominatedness.

Proposition 4.2 *If $n > 2$ and q is a first-best public-good provision rule, there exists no anonymous weakly robustly incentive-compatible direct mechanism that implements q and also satisfies budget balance.*

This proposition parallels a result of Green and Laffont (1979) according to which dominant-strategy implementability of first-best public-good provision rules - and hence robust implementability - is incompatible with budget balance. With weak robustness, rather than robustness, however, the proof is much more involved, a monster.

The argument exploits the fact that, under a weakly robustly incentive-compatible direct mechanism, truthtelling is a Bayes-Nash best-response to truthtelling by other agents under all specifications with unconditionally independent and identically distributed characteristics, in particular, all specifications involving only three values, θ^0, θ^1, θ^2, of the preference parameters, where θ^0, θ^1, θ^2 can be specified so that $\theta^0 = 0 < \theta^1 < \frac{K}{n} < \theta^2 < K$ and $\theta^1 + \theta^2 > K$; with this specification, efficiency requires that the public good be provided if at least one participant has the preference parameter θ^2 and at least one other participant has the preference parameter θ^1 or θ^2.

For any one individual, the characteristics θ^0, θ^1, θ^2 have probabilities π_0, π_1, π_2, and the best-response property of truthtelling must be independent of the values of these probabilities. By considering the neighbourhood of the extreme specification $\pi_0 = \pi_1 = 0$, $\pi_2 = 1$, one finds that, if two or more participants have the preference parameter θ^2, the public good must be provided, and, under anonymity, budget balance, and Bayes-Nash incentive compatibility, all participants must pay $\frac{K}{n}$, i.e., the cost must be shared equally. However, by considering the neighbourhood of the other extreme specification $\pi_0 = 1$, $\pi_1 = \pi_2 = 0$, one also finds, if no participant has the payoff parameter θ^1 and exactly two participants have the payoff parameter θ^2, the payments assigned to these participants cannot exceed θ^1 each, so they do not pay their fair share. The reason is that, in the given situation, any one of the participants with payoff parameter θ^2 must be discouraged from falsely reporting $\hat{\theta} = \theta^1$, which would not affect the level of public-good provision, and in a situation with exactly one participant with payoff parameter θ^2 and exactly one participant with payoff parameter θ^1, the latter participant's payment cannot exceed θ^1 since otherwise this participant would prefer to report $\hat{\theta} = 0$.

5 Robustness, Efficiency, and Approximate Budget Balance for Large n

How serious is the problem of budget imbalances for weakly robustly incentive-compatible mechanisms that implement first-best provision rules? For dominant-strategy-implementation of first-best provision rules, Green and Laffont (1979) have suggested that the problem is unimportant when there are many participants and preference parameters are independent and identically distributed. The reason is that, if there are many participants, then, for any one of them, the probability of being pivotal for the decision on public-good provision is close to zero, so the associated incentive problem is relatively unimportant.

In the following, I apply this reasoning to the present setting. Indexing variables and functions by a superscript n for the number of participants, I assume that, for any n, the random variables $\tilde{\theta}_1^n, ..., \tilde{\theta}_n^n$ are the first n elements of a sequence $\{\tilde{\theta}_j\}_{j=1}^{\infty}$ of random variables such that $\tilde{\theta}_1, \tilde{\theta}_2, ...$ are conditionally independent and identically distributed given some random variable $\tilde{y}$ with values in a separable metric space Y.[13] I also specify the provision cost so that, for some $k > 0$,

$$K^n = k \cdot n \tag{5.1}$$

for all n.

I focus on Clarke-Groves mechanisms, which provide for dominant-strategy implementation without ever running a deficit.[14] For any n, the Clarke-Groves mechanism $(q^n, t_1^n, ..., t_n^n)$ has a provision rule q^n that is first-best and payment functions t_i^n that satisfy

$$t_i^n(\theta_1^n, ..., \theta_n^n) = \left[K^n - \sum_{j \neq i} \theta_j^n\right] \cdot q^n(\theta_1^n, ..., \theta_n^n) + h_i^n(\boldsymbol{\theta}_{-i}), \tag{5.2}$$

where

$$h_i^n(\boldsymbol{\theta}_{-i}) := \max_{\hat{q}} \left[\sum_{j \neq i} \theta_j^n - \frac{n-1}{n} K^n\right] \cdot \hat{q}. \tag{5.3}$$

One easily verifies that this mechanism is implementable in dominant strategies. Therefore it is also robustly incentive-compatible and weakly robustly incentive-compatible.

For given $\theta_1^n, ..., \theta_n^n$, the Clarke-Groves mechanism yields the aggregate budget surplus

$$\begin{aligned} S^n(\theta_1^n, ..., \theta_n^n) &= \sum_{i=1}^n t_i^n(\theta_1^n, ..., \theta_n^n) - K^n \cdot q(\theta_1, ..., \theta_n) \\ &= \sum_{i=1}^n \max_{\hat{q}} \left[\sum_{j \neq i} \theta_j^n - \frac{n-1}{n} K^n\right] \cdot \hat{q} \\ &\quad - \sum_{i=1}^n \left[\sum_{j \neq i} \theta_j^n - \frac{n-1}{n} K^n\right] \cdot q^n(\theta_1^n, ..., \theta_n^n), \end{aligned} \tag{5.4}$$

[13] As mentioned in the introduction, with an infinite sequence $\{\tilde{\theta}_j\}_{j=1}^{\infty}$, this assumption is equivalent to the assumption that the random variables $\tilde{\theta}_1, \tilde{\theta}_2, ...$ are exchangeable.

[14] See Clarke (1971), Groves (1973), Green and Laffont (1979).

which is obviously nonnegative. The surplus is zero if, for every i, $q^n(\theta_1^n, ..., \theta_n^n)$ solves the maximization problem in (5.3). It is positive if, for some i, $q^n(\theta_1, ..., \theta_n)$ fails to solve this maximization problem.

Thus, $S^n(\theta_1^n, ..., \theta_n^n) > 0$ if and only if either

$$q^n(\theta_1^n, ..., \theta_n^n) = 1 \quad \text{and} \quad \sum_{j \neq i} \theta_j < \frac{n-1}{n} K^n \quad \text{for some } i \tag{5.5}$$

or

$$q^n(\theta_1^n, ..., \theta_n^n) = 0 \quad \text{and} \quad \sum_{j \neq i} \theta_j > \frac{n-1}{n} K^n \quad \text{for some } i. \tag{5.6}$$

Given that $q^n(\theta_1^n, ..., \theta_n^n)$ maximizes the welfare indicator $\left[\sum_{j=1}^{n} \theta_j^n - K^n\right] \cdot q$ with respect to q, one obtains:

Lemma 5.1 *For any n and any $(\theta_1^n, ..., \theta_n^n) \in [0, \bar{\theta}]^n$, the aggregate budget surplus $S^n(\theta_1^n, ..., \theta_n^n)$ from the Clarke-Groves mechanism is positive if and only if, for some i, the expressions $\sum_{j \neq i} \theta_j^n - \frac{n-1}{n} K^n$ and $\sum_{j=1}^{n} \theta_j^n - K^n$ have opposite signs.*

For the expressions $\sum_{j \neq i} \theta_j^n - \frac{n-1}{n} K^n$ and $\sum_{j=1}^{n} \theta_j^n - K^n$ to have opposite signs, it must be the case that $\theta_i^n - \frac{1}{n} K^n$ and $\sum_{j=1}^{n} \theta_j^n - K^n$ have the same sign and, moreover,

$$\left| \theta_i^n - \frac{1}{n} K^n \right| > \left| \sum_{j=1}^{n} \theta_j^n - K^n \right|. \tag{5.7}$$

Any agent i for whom (5.7) holds is pivotal for the outcome. For example, if

$$\theta_i^n - \frac{1}{n} K^n > \sum_{j=1}^{n} \theta_j^n - K^n > 0, \tag{5.8}$$

the public good is provided but it would not be provided if agent i reported, say, the preference parameter $\hat{\theta}_i = \frac{1}{n} K^n$; with such a report, the sum of reported valuations would be $\hat{\theta}_i + \sum_{j \neq i} \theta_j^n$, which is less than K^n if $\sum_{j \neq i} \theta_j^n < \frac{n-1}{n} K^n$, as it is if the expressions $\sum_{j \neq i} \theta_j^n - \frac{n-1}{n} K^n$ and $\sum_{j=1}^{n} \theta_j^n - K^n$ have opposite signs. A symmetric consideration applies if the inequalities in (5.8) are reversed.

Lemma 5.1 shows that the Clarke-Groves mechanism earns a surplus if and only if some agent i is pivotal. From (5.2) and (5.3), one finds that, in any situation, non-pivotal agents only pay $\frac{1}{n}K^n \cdot q(\theta_1^n, ..., \theta_n^n)$, their share of the provision cost, and pivotal agents pay more. The payments are

$$t_i^n(\theta_1^n, ..., \theta_n^n) = \frac{1}{n}K^n + \max\left(0, \theta_i^n - \frac{1}{n}K^n - \left|\sum_{j=1}^{n}\theta_j^n - K^n\right|\right) > \frac{1}{n}K^n \quad (5.9)$$

if $q(\theta_1^n, ..., \theta_n^n) = 1$ and

$$t_i^n(\theta_1^n, ..., \theta_n^n) = \max\left(0, -\left(\theta_i^n - \frac{1}{n}K^n\right) - \left|\sum_{j=1}^{n}\theta_j^n - K^n\right|\right) > 0 \quad (5.10)$$

if $q(\theta_1^n, ..., \theta_n^n) = 0$. Surpluses rest entirely on payments from pivotal agents.

I claim that, if n is large, it is unlikely for an individual to be pivotal. For the specification $K^n = k \cdot n$, (5.7) can be rewritten as

$$|\theta_i^n - k| > \left|\sum_{j=1}^{n}(\theta_j^n - k)\right|. \quad (5.11)$$

Since θ_i^n belongs to the interval $[0, \bar{\theta}]$, (5.11) requires that

$$\bar{\theta} + k > \left|\sum_{j=1}^{n}(\theta_j^n - k)\right|, \quad (5.12)$$

so that the sum $\sum_{j=1}^{n}(\theta_j^n - k)$ belongs to the compact interval $[-(\bar{\theta} + k), \bar{\theta} + k]$. For large n, however, $\sum_{j=1}^{n}(\theta_j^n - k)$ is unlikely to stay bounded in absolute value. This observation yields

Proposition 5.2 *As n goes out of bounds, the aggregate budget surplus from the Clarke-Groves mechanism in the model with n participants and provision cost $K^n = k \cdot n$ converges to zero in probability.*

The following corollary to Proposition 5.2 recovers the main conclusion of Green and Laffont (1979) on this subject.

Corollary 5.3 *As n goes out of bounds, the expected value of aggregate budget surplus per capita from the Clarke-Groves mechanism in the model with n participants and provision cost $K^n = k \cdot n$ converges to zero.*

For the individual participants' payments, the same argument also yields:

Corollary 5.4 *As n goes out of bounds, for any i, the excess $t_i^n(\tilde{\theta}_1^n, ..., \tilde{\theta}_n^n) - kq^n(\tilde{\theta}_1^n, ..., \tilde{\theta}_n^n)$ of the payment of agent i under the Clarke-Groves mechanism in the model with n participants and provision cost $K^n = k \cdot n$ over the share of agent i in the provision cost converges to zero in probability, and the expected payment $Et_i^n(\tilde{\theta}_1^n, ..., \tilde{\theta}_n^n)$ converges to the provision cost k.*

6 Robust First-Best Implementation with Budget Balance in a Large Population

The preceding results suggest that the problem of public-good provision is easiest to analyse when the population is large and no one agent is pivotal. Because no one agent is pivotal, any direct mechanism is robustly incentive-compatible.[15] In particular, a mechanism stipulating that the public good be provided if and only if the cross-section average valuation exceeds the *per-capita* provision cost k, with payment functions requiring agents to pay equal shares of the cost, is robustly incentive-compatible and implements a first-best provision rule. According to the preceding results, such a mechanism can be interpreted as a limit of Clarke-Groves mechanisms for large finite populations.

Large-population models of public-good provision have not been much studied, perhaps because of technical issues. With a large population, the efficiency condition in the preceding paragraph depends on the comparison of the per-capita provision cost k with the average

$$\int_A \tilde{\theta}(\omega, a) d\alpha(a), \tag{6.1}$$

A is the space of agents and α is a measure on A. The preceding paragraph presumes that, in the large population, the cross-section average (6.1) is always well defined. Whether this presumption is justified depends on how the domain of the measure α is specified.

For example, if A is the Lebesgue unit interval and, conditionally on the aggregate shock $\tilde{y}$, the random variables $\tilde{\theta}(\cdot, a)$, $a \in A$, are independent, then either the integral in (6.1) is almost surely undefined or, conditionally on $\tilde{y}$, the random variables $\tilde{\theta}(\cdot, a)$, $a \in A$, are degenerate so that $\tilde{\theta}(\cdot, a) = \theta^*(\tilde{y}(\cdot))$ with probability one.[16] This dilemma can be avoided, however, if the algebra of measurable subsets of A and the algebra of measurable subsets of $\Omega \times A$ are suitably enlarged.[17] When this is done, the cross-section average (6.1) is well

[15] See Proposition 1 in Hellwig (2021).

[16] Hammond and Sun (2008), Qiao et al. (2016).

[17] For details, see Sun (2006), Qiao et al. (2016), Hellwig (forthcoming). The σ-algebra on $\Omega \times A$ must be a *Fubini extension* of the standard product σ-algebra, so that, for any bounded measurable function f on $\Omega \times A$, the mappings

$$\omega \to \int_A f(\omega, a) d\alpha(a) \quad \text{and} \quad a \to \int_\Omega f(\omega, a) dP(\omega)$$

defined with probability one. Moreover, if, for α-almost all a and a' in A, the random variables $\tilde{\theta}(\cdot, a)$ and $\tilde{\theta}(\cdot, a')$ are conditionally independent given $\tilde{y}$, a conditional exact law of large numbers ensures that

$$\int_A \tilde{\theta}(\omega, a) d\alpha(a) = \theta^*(\tilde{y}(\omega)) \tag{6.2}$$

with probability one where, as before $\theta^*(\tilde{y})$ is the common value of the conditional expectation $E[\tilde{\theta}(\cdot, a)|\tilde{y}]$.[18] Thus a decision rule driven by the comparison of (6.1) with k effectively turns on the comparison of the conditional expectation $\theta^*(\tilde{y}(\cdot))$ with k.

In the large-economy version of the model, individual uncertainty plays no role. Information about the macro variable $\tilde{y}$ and about $\theta^*(\tilde{y})$ is all that matters for efficient public-good provision. In the absence of coalitions coordinating their members' behaviours, there is no problem in obtaining this information through a robustly individually incentive-compatible direct mechanism with budget balance.[19]

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are well-defined and measurable and, moreover, the Fubini equation

$$\int_\Omega \int_A f(\omega, a) d\alpha(a) dP(\omega) = \int_A \int_\Omega f(\omega, a) dP(\omega) d\alpha(a)$$

holds. Moreover, it must be *rich*, i.e., there must exist a measurable function h from $\Omega \times A$ to the unit interval such that, for α-almost all $a \in A$, the random variable $h(\cdot, a)$ has a uniform distribution and, for α-almost all a and a', the random variables $h(\cdot, a)$ and $h(\cdot, a')$ are independent. Existence of a rich Fubini extension is compatible with $A = [0, 1]$ provided the algebra of measurable subsets of A is suitably enlarged, relative to the Lebesgue σ.algebra. See Sun and Zhang (2009).

[18] See Qiao et al. (2016).

[19] Bierbrauer and Hellwig (2015) and Hellwig (2021) show that, whereas in a large population robust individual incentive compatibility is trivially satisfied, a requirement of robust coalition incentice compatibility imposes serious constraints, not on the information that can be obtaines, but on the uses that can be made of this information.

A Proofs

Proposition 2.1 needs no proof because it merely restates the cited result of Güth and Hellwig (1986)

Proof of Proposition 2.2. Let $(q, t_1, ..., t_n)$ be as specified in the proposition. By (2.12), the function Q_i that is defined by (2.5) satisfies (2.14). By (2.10),

$$T_i(\hat{\theta}_i|\hat{y}) := \int_{[0,\bar{\theta}]^{n-1}} t_i(\hat{\theta}_i, \boldsymbol{\theta}_{-i}|\hat{y})\ dF^{n-1}(\boldsymbol{\theta}_{-i})$$

for all i, $\hat{\theta}_i$, and $\hat{y}$, so the function T_i that is defined by (2.6) satisfies (2.15). The expected net payoff (2.4) that agent i obtains from reporting $\hat{\theta}_i$ when the true preference parameter is θ_i is therefore equal to

$$\theta_i \cdot Q_i(\hat{\theta}_i, \theta_i) - T_i(\hat{\theta}_i, \theta_i) = \int_Y \left[\theta_i \cdot Q_i(\hat{\theta}_i|\hat{y}) - T_i(\hat{\theta}_i|\hat{y})\right]\ dG(\hat{y}|\theta_i). \tag{A.1}$$

Incentive compatibility follows because, for any $\hat{y} \in Y$, the integrand in (A.1) is maximized by setting $\hat{\theta}_i = \theta_i$. ■

Proof of Proposition 3.1. The first statement of the proposition follows from Proposition 2.2 and the observation that any first-best provision rule q is nondecreasing in $\theta_1, ..., \theta_n$. For the second statement, use (2.13) to obtain

$$\sum_{i=1}^{n} t_i(\theta_1, ..., \theta_n) = \int_Y \sum_{i=1}^{n} t_i(\theta_1, ..., \theta_n|\hat{y})\ dG(\hat{y}). \tag{A.2}$$

For any $\hat{y} \in Y$, (2.10) implies

$$\begin{aligned}
& \sum_{i=1}^{n} t_i(\theta_1, ..., \theta_n|\hat{y}) \\
= \;& \sum_{i=1}^{n} T_i(\theta_i|\hat{y}) + K \cdot q(\theta_1, ..., \theta_n) - \frac{1}{n} \sum_{i=1}^{n} \int K \cdot q(\theta_i, \hat{\boldsymbol{\theta}}_{-i}))\ dF^{n-1}(\hat{\boldsymbol{\theta}}_{-i}|\hat{y}) \\
& -\frac{1}{n-1} \sum_{i=1}^{n} \sum_{j \neq i} \left[T_j(\theta_j|\hat{y}) - \int T_j(\theta'_j|\hat{y})\ dF(\theta'_j|\hat{y})\right] \\
& +\frac{1}{n-1} \sum_{i=1}^{n} \sum_{j \neq i} \frac{1}{n} \left[\int K \cdot (q(\theta_j, \boldsymbol{\theta}'_{-j})\ dF^{n-1}(\boldsymbol{\theta}'_{-j}|\hat{y}) - \int K \cdot q(\hat{\boldsymbol{\theta}})\ dF^{n}(\hat{\boldsymbol{\theta}}|\hat{y})\right] \\
= \;& K \cdot q(\theta_1, ..., \theta_n) + \sum_{j=1}^{n} \int T_j(\theta'_j|\hat{y})\ dF(\theta'_j|\hat{y}) - \int K \cdot q(\hat{\boldsymbol{\theta}})\ dF^{n}(\hat{\boldsymbol{\theta}}|\hat{y}).
\end{aligned} \tag{A.3}$$

The budget balance condition (3.2) is satisfied if the constants $T_j(0|\hat{y})$ are arranged so that

$$\sum_{j=1}^{n} T_j(0|\hat{y}) + \sum_{j=1}^{n} \int [T_j(\theta'_j|\hat{y}) - T_j(0|\hat{y})]\ dF(\theta'_j|\hat{y}) = \int K \cdot q(\hat{\boldsymbol{\theta}})\ dF^{n}(\hat{\boldsymbol{\theta}}|\hat{y}) \tag{A.4}$$

for all $\hat{y} \in Y$, taking account of the fact that the differences $T_j(\theta_j'|\hat{y}) - T_j(0|\hat{y})$ are determined by (2.11) and the provision rule q. ■

Proof of Lemma 4.1. Suppose that the direct mechanism $(q, t_1, ..., t_n)$ is *weakly robustly incentive-compatible.* Let Φ be any prior of the form $\Phi = (F_\Phi)^n$. Let Y, G, and the mapping $y \longmapsto F(\cdot|y)$ be such that $F(\cdot|y) = F_\Phi$ for all y. Then trivially $(q, t_1, ..., t_n)$ is Bayesian incentive-compatible for Φ.

Conversely, suppose that $(q, t_1, ..., t_n)$ is Bayesian incentive-compatible for any prior taking the form F^n. Then, trivially, $(q, t_1, ..., t_n)$ is also Bayesian incentive-compatible for any prior taking the form $\int_Y F^n(\cdot|y)dG(y)$, for any Y, any G, and any mapping $y \longmapsto F(\cdot|y)$ from Y to the space of probability measures on $[0, \bar{\theta}]$. ■

Proof of Proposition 4.2. The proof proceeds indirectly. Suppose that $(q, t_1, ..., t_n)$ is an anonymous *weakly robustly incentive-compatible* direct mechanism that implements q and also satisfies the balanced-budget condition (3.2) for all $(\theta_1, ..., \theta_n) \in [0, \bar{\theta}]^n$. With an abuse of notation, I write $q(\hat{\theta}|D)$ and $t(\hat{\theta}, D)$ for the public-good provision level and the contribution made by a particular agent if that agent announces the value $\hat{\theta}$ of his preference parameter and the distribution of the *other* agents' preference parameters is equal to D. Notice that the preference parameter of the agent in question does not enter the calculation of the distribution D.

Lemma 4.1 implies that, for every probability measure F on $[0, \bar{\theta}]$, the mechanism $(q, t_1, ..., t_n)$ is Bayesian incentive-compatible if the participants' characteristics are independent and identically distributed with the common distribution F. I consider the implications of this statement for probability distributions F whose support is set $\{\theta^0, \theta^1, \theta^2\}$ such that

$$\theta^0 = 0, \quad \theta^1 = \frac{1}{2n} \cdot K, \quad \theta^2 = K - \frac{1}{4n} \cdot K. \tag{A.5}$$

Any such probability distribution is characterized by the probabilities π_0, π_1, π_2 that it assigns to the points $\theta^0, \theta^1, \theta^2$, where $\pi_2 = 1 - \pi_0 - \pi_1$. Similarly, any cross-section distribution D of preference parameters of other agents is characterized by the numbers m_0, m_1, m_2 of other agents that have preference parameters $\theta^0, \theta^1, \theta^2$.

Given an arbitrary triple (π_0, π_1, π_2) of probabilities assigned to the points $\theta^0, \theta^1, \theta^2$distribution, Bayesian incentive compatibility requires that, for any θ^i, the report $\hat{\theta}(\theta^i) = \theta^i$ maximizes the objective

$$\sum_{m_0, m_1, m_2} \frac{(n-1)!}{m_0! m_1! m_2!} \cdot \pi_0^{m_0} \pi_1^{m_1} \pi_2^{m_2} \cdot \left[\theta^i \cdot q(\hat{\theta}|m_0, m_1, m_2) - t(\hat{\theta}|m_0, m_1, m_2)\right], \tag{A.6}$$

where the sum is taken over all triples (m_0, m_1, m_2) such that m_0, m_1, m_2 add up to $n - 1$, the number of other agents.

By (3.1), the assumption that q is a first-best provision rule implies that

$$q(\hat{\theta}|m_0, m_1, m_2) = 1 \text{ if } \hat{\theta} + (\theta^1 \cdot m_1 + \theta^2 \cdot m_2) \cdot K > K \tag{A.7}$$

and

$$q(\hat{\theta}|m_0, m_1, m_2) = 0 \;\text{ if }\; \hat{\theta} + (\theta^1 \cdot m_1 + \theta^1 \cdot m_2) \cdot K < K. \tag{A.8}$$

In particular, by (A.5) and anonymity,

$$q(\theta^0|n-3, 0, 2) = q(\theta^2|n-2, 0, 1) = 1. \tag{A.9}$$

For this constellation, budget balance requires that

$$(n-2) \cdot t(\theta^0|n-3, 0, 2) + 2 \cdot t(\theta^2|n-2, 0, 1) = K. \tag{A.10}$$

I will however prove that

$$t(\theta^0|n-3, 0, 2) = \frac{K}{n} \tag{A.11}$$

and

$$t(\theta^2|n-2, 0, 1) \leq \theta^1. \tag{A.12}$$

By (A.5), (A.11) and (A.12) imply

$$(n-2) \cdot t(\theta^0|n-3, 0, 2) + 2 \cdot t(\theta^2|n-2, 0, 1) \leq \frac{n-1}{n} \cdot K, \tag{A.13}$$

so that the budget-balance condition (A.10) cannot hold.

I give separate arguments for the inequalities (A.11) and (A.12).

Proof of (A.11). I will prove the more general claim that, if $n > 2$, then, for any $r \in \{0, ..., n-3\}$ and any $\hat{\theta} \in \{\theta^0, \theta^1, \theta^2\}$,

$$q(\hat{\theta}|r, 0, n-1-r) = 1 \tag{A.14}$$

and

$$t(\hat{\theta}|r, 0, n-1-r) = \frac{K}{n}. \tag{A.15}$$

From this more general claim, (A.11) follows because this equation is a special case of (A.15), with $r = n-3$ and $\hat{\theta} = \theta^0$.

To prove that (A.14) must hold for all $r \in \{0, ..., n-3\}$ and all $\hat{\theta} \in \{\theta^0, \theta^1, \theta^2\}$, it suffices to observe that, for $\hat{\theta} \in \{\theta^0, \theta^1, \theta^2\}$, $m_0 = r, m_1 = 0$, and $m_2 = n-1-r$,

$$\hat{\theta} + \theta^1 m_1 + \theta^2 m_2 \geq \theta^2(n-1-(n-3)) \geq \theta^2 \cdot 2 > K,$$

so (A.14) follows from (A.7).

I next prove that (A.15) must hold for all $r \in \{0, ..., n-3\}$ and all $\hat{\theta} \in \{\theta^0, \theta^1, \theta^2\}$. Weak robustness requires that, for any θ^i, the report $\hat{\theta} = \theta^i$ maximizes the expression (A.6), regardless of the probabilities π_0, π_1, π_2. In particular, this report maximizes (A.6) if $\pi_1 = 0$, regardless of π_0 and π_2. For $\pi_1 = 0$, (A.6) takes the form

$$\sum_{m_0, m_2} \frac{(n-1)!}{m_0! m_2!} \cdot \pi_0^{m_0} \pi_2^{m_2} \cdot \left[\theta^i \cdot q(\hat{\theta}|m_0, 0, m_2) - t(\hat{\theta}|m_0, 0, m_2)\right]. \tag{A.16}$$

I now proceed by induction on r, beginning with $r = 0$. By weak robustness, for any θ^i, the report $\hat{\theta} = \theta^i$ maximizes the expression (A.16), regardless of π_0 and π_2. In particular, this report maximizes (A.16) when $\pi_0 = 0$ and $\pi_2 = 1$. In this case, (A.16) takes the form

$$\theta^i \cdot q(\hat{\theta}|0, 0, n-1) - t(\hat{\theta}|0, 0, n-1) = \theta^i - t(\hat{\theta}|0, 0, n-1), \tag{A.17}$$

where the last equation follows from (A.14). For $\hat{\theta} = \theta^i$ to maximize (A.17) for all $\theta^i \in \{\theta^0, \theta^1, \theta^2\}$, it must be the case that $t(\hat{\theta}|0, 0, n-1)$ is independent of $\hat{\theta}$. Moreover, by anonymity and budget balance, $t(\theta^2|0, 0, n-1) = \frac{K}{n}$. Thus,

$$t(\hat{\theta}|0, 0, n-1) = \frac{K}{n} \quad \text{for all} \quad \hat{\theta} \in \{\theta^0, \theta^1, \theta^2\}, \tag{A.18}$$

which is just (A.15) for $r = 0$.

For the induction step of the argument, suppose that (A.15) has been shown to be true for $r' \in \{0, ..., r\}$. Then in (A.16), the terms involving $m_0 \in \{0, ..., r\}$, and $m_2 = n - 1 - m_0$ are all independent of $\hat{\theta}$, being proportional to $\theta^i - \frac{K}{n}$. Therefore, these terms play no role in the maximization of (A.16), i.e., for any θ^i and any π_0 and $\pi_2 = 1 - \pi_0$, a maximizer $\hat{\theta}(\theta_i)$ of (A.16) is also a maximizer of

$$\sum_{\substack{m_0 > r, \\ m_2 = n-1-m_0}} \frac{(n-1)!}{m_0! m_2!} \cdot \pi_0^{m_0} \pi_2^{m_2} \cdot \left[\theta^i \cdot q(\hat{\theta}|m_0, 0, m_2) - t(\hat{\theta}|m_0, 0, m_2)\right]. \tag{A.19}$$

If $\pi_0 > 0$ and $\pi_1 = 0$, maximization of (A.19) in turn is equivalent to the maximization of

$$\sum_{\substack{m_0 > r, \\ m_2 = n-1-m_0}} \frac{(n-1)!}{m_0! m_2!} \cdot \pi_0^{m_0 - 1} \pi_2^{m_2} \cdot \left[\theta^i \cdot q(\hat{\theta}|m_0, 0, m_2) - t(\hat{\theta}|m_0, 0, m_2)\right]. \tag{A.20}$$

For $\pi_0 = \Delta$ and $\pi_2 = 1 - \Delta$, if $\Delta \downarrow 0$, then, for any θ^i and any $\hat{\theta}$ in $\{\theta^0, \theta^1, \theta^2\}$, expression (A.20) converges to

$$\frac{(n-1)!}{r+1!(n-r-2)!} \cdot [\theta^i \cdot q(\hat{\theta}|r+1, 0, n-r-2) - t(\hat{\theta}|r+1, 0, n-r-2)]. \tag{A.21}$$

By the maximum theorem, the assumption that, regardless of $\theta^i \in \{\theta^0, \theta^1, \theta^2\}$, setting $\hat{\theta}(\theta^i) = \theta^i$ maximizes (A.20) for $\pi_0 = \Delta > 0$ and $\pi_2 = 1 - \Delta < 1$ implies that, regardless of $\theta^i \in \{\theta^0, \theta^1, \theta^2\}$, setting $\hat{\theta}(\theta^i) = \theta^i$ maximizes (A.21), the limit of (A.20) as $\Delta \downarrow 0$.

Along the same lines as before, I note that, since $n > r$, (A.5) and (A.7) imply that $q(\hat{\theta}|r+1, 0, n-r-2) = 1$, regardless of $\hat{\theta} \in \{\theta^0, \theta^1, \theta^2\}$. For $\hat{\theta} = \theta^i$ to maximize (A.21) for all $\theta^i \in \{\theta^0, \theta^1, \theta^2\}$, it must therefore be the case that $t(\hat{\theta}|r+1, 0, n-r-2)$ is independent of $\hat{\theta}$. Moreover, by anonymity and budget balance,$t(\theta^2|r+1, 0, n-r-2) = \frac{K}{n}$. Thus,

$$t(\hat{\theta}|r+1, 0, n-r-2) = \frac{K}{n} \quad \text{for all} \quad \hat{\theta} \in \{\theta^0, \theta^1, \theta^2\}, \tag{A.22}$$

and the induction is complete.

Proof of (A.12). I first note that, for $\pi_0 = 1$, $\pi_1 = \pi_2 = 0$, expression (A.6) takes the form

$$\theta^i \cdot q(\hat{\theta}|n-1,0,0) - t(\hat{\theta}|n-1,0,0). \tag{A.23}$$

For $m_0 = n-1$ and $m_1 = m_2 = 0$,

$$\hat{\theta} + \theta^1 m_1 + \theta^2 m_2 = \hat{\theta} < K,$$

regardless of $\hat{\theta}$ so (A.8) implies

$$q(\hat{\theta}|n-1,0,0) = 0 \tag{A.24}$$

for all $\hat{\theta} \in \{\theta^0, \theta^1, \theta^2\}$. Thus,

$$\theta^i \cdot q(\hat{\theta}|n-1,0,0) - t(\hat{\theta}|n-1,0,0) = -t(\hat{\theta}|n-1,0,0). \tag{A.25}$$

For $\hat{\theta}(\theta^i) = \theta^i$ to maximize (A.23), for all $\theta^i \in \{\theta^0, \theta^1, \theta^2\}$, it must therefore be the case that $t(\hat{\theta}|n-1,0,0)$ is independent of $\hat{\theta}$. By anonymity and budget balance, $t(\theta^0|n-1,0,0) = 0$. Therefore

$$t(\hat{\theta}|n-1,0,0) = 0 \tag{A.26}$$

for all $\hat{\theta} \in \{\theta^0, \theta^1, \theta^2\}$.

Thus, expression (A.23) is equal to zero, regardless of $\hat{\theta}$. Expression (A.6) can therefore be rewritten as

$$\sum_{m_0 < n-1, m_1, m_2} \frac{(n-1)!}{m_0! m_1! m_2!} \cdot \pi_0^{m_0} \pi_1^{m_1} \pi_2^{m_2} \cdot \left[\theta^i \cdot q(\hat{\theta}|m_0, m_1, m_2) - t(\hat{\theta}|m_0, m_1, m_2)\right], \tag{A.27}$$

where the sum is taken over all $m_0 < n-1, m_1$, and m_2 such that m_0, m_1, m_2 add up to $n-1$. For $\pi_2 > 0$, regardless of θ^i, expression (A.27) is maximized by the same $\hat{\theta}$ as the expression

$$\sum_{m_0 < n-1, m_1, m_2} \frac{(n-1)!}{m_0! m_1! m_2!} \cdot \pi_0^{m_0} \pi_1^{m_1} \pi_2^{m_2 - 1} \cdot \left[\theta^i \cdot q(\hat{\theta}|m_0, m_1, m_2) - t(\hat{\theta}|m_0, m_1, m_2)\right] \tag{A.28}$$

For $\pi_0 = 1 - \Delta_1 - \Delta_2, \pi_1 = \Delta_1 > 0$ and $\pi_2 = \Delta_2 > 0$, if $\Delta_1 \downarrow 0$ and $\Delta_2 \downarrow 0$, then, for any θ^i and any $\hat{\theta}$ in $\{\theta^0, \theta^1, \theta^2\}$, expression (A.28) converges to

$$(n-1) \cdot [\theta^i \cdot q(\hat{\theta}|n-2,0,1) - t(\hat{\theta}|n-2,0,1)]. \tag{A.29}$$

By the maximum theorem, the assumption that, regardless of $\theta^i \in \{\theta^0, \theta^1, \theta^2\}$, setting $\hat{\theta}(\theta^i) = \theta^i$ maximizes (A.28) for $\pi_0 = 1 - \Delta_1 - \Delta_2, \pi_1 = \Delta_1 > 0$ and $\pi_2 = \Delta_2 > 0$ implies that, regardless of $\theta^i \in \{\theta^0, \theta^1, \theta^2\}$, setting $\hat{\theta}(\theta^i) = \theta^i$ maximizes (A.29), the limit of (A.28) as $\Delta_1 \downarrow 0$ and $\Delta_2 \downarrow 0$.

By (A.8) and (A.7), in combination with (A.5),

$$q(\hat{\theta}|n-2,0,1) = 0 \;\text{ if }\; \hat{\theta} = \theta^0 \tag{A.30}$$

and

$$q(\hat{\theta}|n-2,0,1) = 1 \;\text{ if }\; \hat{\theta} \in \{\theta^1, \theta^2\}. \tag{A.31}$$

By anonymity and (A.26), it must also be the case that

$$t(\theta^0|n-2,0,1) = t(\theta^2|n-1,0,0) = 0, \tag{A.32}$$

so for $\hat{\theta} = 0$, the maximand (A.29) is equal to zero, regardless of θ^i.

For $\hat{\theta}(\theta^i) = \theta^i$ to be maximizing (A.29) when $\theta^i = \theta^1$ and $\theta^i = \theta^2$, it must be the case that

$$\theta^1 \cdot q(\theta^1|n-2,0,1) - t(\theta^1|n-2,0,1) \geq \theta^1 \cdot q(\theta^0|n-2,0,1) - t(\theta^0|n-2,0,1) \tag{A.33}$$

and

$$\theta^2 \cdot q(\theta^2|n-2,0,1) - t(\theta^2|n-2,0,1) \geq \theta^2 \cdot q(\theta^1|n-2,0,1) - t(\theta^1|n-2,0,1). \tag{A.34}$$

Upon combining (A.33) with (A.30) and (A.32), one obtains

$$t(\theta^1|n-2,0,1) \leq \theta^1. \tag{A.35}$$

Upon combining (A.34) with (A.35), one obtains (A.12). This completes the proof of Proposition 4.2. ■

Proof of Proposition 5.2. Let $(\Omega, \mathcal{F}, P)$, with generic element ω, be the underlying probability space on which all random variables are defined. The proposition claims that, as $n \to \infty$, the probability of the event

$$\{\omega \in \Omega |\ \tilde{S}^n(\omega) > 0\}$$

goes to zero, where

$$\tilde{S}^n := S^n(\tilde{\theta}_1, ..., \tilde{\theta}_n). \tag{A.36}$$

By Lemma 5.1 and (5.12),

$$\{\omega \in \Omega |\ \tilde{S}^n(\omega) > 0\} \subset \left\{ \omega \in \Omega |\ \left| \sum_{j=1}^{n} (\tilde{\theta}_j(\omega) - k) \right| < \bar{\theta} + k \right\}.$$

Thus it suffices to show that, as $n \to \infty$,

$$P\left(\left\{ \omega \in \Omega |\ \left| \sum_{j=1}^{n} (\tilde{\theta}_j(\omega) - k) \right| < \bar{\theta} + k \right\} \right) \to 0. \tag{A.37}$$

Let $\theta^*(\tilde{y})$ be the common conditional expectation of the random variables $\tilde{\theta}_1, \tilde{\theta}_2, ...$ given $\tilde{y}$. I will give separate arguments for the case where $\theta^*(\tilde{y}) \neq k$

with probability one, the case where $\theta^*(\tilde{y}) = k$ with probability one, and the case where both events, $\theta^*(\tilde{y}) \neq k$ and $\theta^*(\tilde{y}) = k$, have positive probabilities.

Case 1: $\theta^*(\tilde{y}) \neq k$ with probability one.

The sequence $\tilde{\theta}_1, \tilde{\theta}_2, ...$ satisfies the conditions for the strong law of large numbers for conditionally independent and identically distributed random variables, as stated in Beck (1974). As $n \to \infty$, therefore

$$\frac{1}{n}\sum_{j=1}^{n}[\tilde{\theta}_j - \theta^*(\tilde{y})] \to 0, \quad \text{almost surely,} \tag{A.38}$$

and hence

$$\frac{1}{n}\sum_{j=1}^{n}[\tilde{\theta}_j - k] \to \theta^*(\tilde{y}) - k, \quad \text{almost surely.} \tag{A.39}$$

If $\theta^*(\tilde{y}) \neq k$ with probability one, it follows that

$$\left|\sum_{j=1}^{n}(\tilde{\theta}_j(\omega) - k)\right| \to \infty, \quad \text{almost surely,}$$

and, hence, that, for almost every $\omega \in \Omega$, there exists $N(\omega)$ such that, for $n > N(\omega)$,

$$\left|\sum_{j=1}^{n}(\tilde{\theta}_j(\omega) - k)\right| > \bar{\theta} + k.$$

(A.37) follows immediately.[20]

Case 2: $\theta^*(\tilde{y}) = k$ with probability one.

The sequence $\tilde{\theta}_1, \tilde{\theta}_2, ...$ satisfies the conditions for the central limit theorem for conditionally independent and identically distributed random variables, as stated in Yuan et al. (2014). Therefore there exists a normal distributed random variable $\tilde{X}$ with $E\tilde{X} = 0$, such that, as $n \to \infty$,

$$\frac{1}{n^{\frac{1}{2}}}\sum_{j=1}^{n}[\tilde{\theta}_j - \theta^*(\tilde{y})] \to \tilde{X} \quad \text{in distribution.} \tag{A.40}$$

Thus, for any $\Delta > 0$,

$$P\left(\left\{\omega \in \Omega \middle| \left|\frac{1}{n^{\frac{1}{2}}}\sum_{j=1}^{n}(\tilde{\theta}_j(\omega) - k)\right| < \Delta\right\}\right) \to \Phi((-\Delta, +\Delta)),$$

[20] For the case $\theta^*(\tilde{y}) \neq k$ with probability one, the argument given actually shows that $\tilde{S}^n \to 0$ with probability one as $n \to \infty$.

as $n \to \infty$, where Φ is the measure associated with the normal distribution. Notice that

$$P\left(\left\{\omega \in \Omega | \left|\sum_{j=1}^{n}(\tilde{\theta}_j(\omega) - k)\right| < \bar{\theta} + k\right\}\right)$$
$$= P\left(\left\{\omega \in \Omega | \left|\frac{1}{n^{\frac{1}{2}}}\sum_{j=1}^{n}(\tilde{\theta}_j(\omega) - k)\right| < \frac{1}{n^{\frac{1}{2}}}(\bar{\theta} + k)\right\}\right)$$

and that, for any $\Delta > 0$ and any sufficiently large n,

$$\frac{1}{n^{\frac{1}{2}}}(\bar{\theta} + k) < \Delta.$$

For any $\Delta > 0$, therefore,

$$\limsup P\left(\left\{\omega \in \Omega | \left|\sum_{j=1}^{n}(\tilde{\theta}_j(\omega) - k)\right| < \bar{\theta} + k\right\}\right) \leq \Phi((-\Delta, +\Delta)).$$

Upon taking limits as $\Delta \downarrow 0$, one obtains

$$\limsup P\left(\left\{\omega \in \Omega | \left|\sum_{j=1}^{n}(\tilde{\theta}_j(\omega) - k)\right| < \bar{\theta} + k\right\}\right) = 0.$$

Again, (A.37) follows immediately.

Case 3: $\theta^*(\tilde{y}) \neq k$ **with positive probability and** $\theta^*(\tilde{y}) = k$ **with positive probability.**

To conclude the proof, consider the general case, where both events, $\theta^*(\tilde{y}) \neq k$ and $\theta^*(\tilde{y}) = k$, have positive probabilities. Given the probability distribution G of the random variable $\tilde{y}$, let $G_{\neq}$ and $G_{=}$ be regular conditional distributions for $\tilde{y}$ conditional on the events $\theta^*(\tilde{y}) \neq k$ and $\theta^*(\tilde{y}) = k$. Define random variables $\tilde{y}_{\neq}$ and $\tilde{y}_{=}$ on $(\Omega, \mathcal{F}, P)$ such that $\tilde{y}_{\neq}$ has the distribution $G_{\neq}$ and $\tilde{y}_{=}$ has the distribution $G_{=}$. Define preference parameter processes, $\tilde{\theta}_1^{\neq}, \tilde{\theta}_2^{\neq}, ...$ and $\tilde{\theta}_1^{=}, \tilde{\theta}_2^{=}, ...$, such that the random variables $\tilde{\theta}_1^{\neq}, \tilde{\theta}_2^{\neq}, ...$ are conditionally independent and identically distributed given $\tilde{y}_{\neq}$, the random variables $\tilde{\theta}_1^{=}, \tilde{\theta}_2^{=}, ...$ are conditionally independent and identically distributed given $\tilde{y}_{=}$, and moreover, for any $y \in Y$ and $j = 1, 2, ...$, the probability distribution of $\tilde{\theta}_j^{\neq}$ conditional on the event $\tilde{y}_{\neq} = y$ or of $\tilde{\theta}_j^{=}$ conditional on the event $\tilde{y}_{=} = y$ is given by $F(\cdot|y)$. The random variables $\tilde{y}_{\neq}, \tilde{\theta}_1^{\neq}, \tilde{\theta}_2^{\neq}, ...$ define an incomplete-information model with $\theta^*(\tilde{y}_{\neq}) \neq k$ almost surely, and the random variables $\tilde{y}_{=}, \tilde{\theta}_1^{=}, \tilde{\theta}_2^{=}, ...$ define an incomplete-information model with $\theta^*(\tilde{y}_{\neq}) \neq k$ almost surely. The arguments for Cases 1 and 2 imply that in each of these models, the aggregate budget surplus from the Clarke-Groves mechanism converges to zero in probability as

n goes out of bounds. The desired result follows from the observation that the probability distribution of the aggregate budget surplus in the original model is a mixture, with weights $G(\{y \in Y|\theta^*(y) \neq k\})$ and $G(\{y \in Y|\theta^*(y) = k\})$, of the probability distributions of the aggregate budget surpluses in the models with random variables $\tilde{y}_{\neq}, \tilde{\theta}_1^{\neq}, \tilde{\theta}_2^{\neq}, ...$ and with random variables $\tilde{y}_{=}, \tilde{\theta}_1^{=}, \tilde{\theta}_2^{=}, ...$ ■

Proof of Corollary 5.3. By Proposition 5.2 and Lebesgue's Bounded Convergence Theorem, it suffices to show that $\frac{1}{n} \cdot \tilde{S}^n$ is uniformly bounded. From (5.9) and (5.10), one obtains

$$\tilde{S}^n = \sum_{i=1}^{n} \max\left(0, \tilde{\theta}_i^n - k - \left|\sum_{j=1}^{n} (\theta_j^n - k)\right|\right)$$

if $q^n(\tilde{\theta}_1^n, ..., \theta_n^n) = 1$ and

$$\tilde{S}^n = \sum_{i=1}^{n} \max\left(0, -(\theta_i^n - k) - \left|\sum_{j=1}^{n} (\theta_j^n - k)\right|\right)$$

if $q^n(\tilde{\theta}_1^n, ..., \theta_n^n) = 0$. In either case,

$$\tilde{S}^n \leq \sum_{i=1}^{n} \max\left(0, \tilde{\theta}_i^n - k\right) \leq n \cdot (\bar{\theta} + k),$$

hence $\frac{1}{n} \cdot \tilde{S}^n \leq \bar{\theta} + k$. ■

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