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A Homeomorphism Theorem for the Universal Type Space with the Uniform Topology*

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Abstract

The paper proves a homeomorphism theorem for the universal type space with the uniform strategic topology or the uniform weak topology that Dekel et al. (2006) and Chen et al. (2010) introduced in order to take account of the fact that beliefs of arbitrarily high orders in agents' belief hierarchies can have a significant impact on strategic behaviour. Probability measures on the larger σ-algebras associated with these finer topologies are the completions of probability measures on the product σ.algebras, Kolmogorov's extension theorem can still be used to derive probability measures from belief hierarchies. The extended Kolmogorov mapping that is thus obtained is a homeomorphism. The paper shows that the canonical mapping from abstract type spaces with continuous belief functions to the universal type space with the uniform topology is continuous if, in the abstract type space setting, beliefs have the topology induced by the total-variation metric.

Key Words: incomplete information, universal type space, uniform weak topology, uniform strategic topology, homeomorphism theorem.

JEL Classification: C70, C72.

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1 Introduction

In the theory of strategic games with incomplete information, the *universal type space* plays a cantreal role because permits a complete representation of *all* strategically relevant aspects of incomplete information in a given setting. In one formalization of this space, an agent's "type" is given by an infinite hierarchy of beliefs about the exogenous parameters affecting the strategic situation, about the other agents' beliefs about the exogenous parameters, about the other agents' beliefs about other agents' beliefs about the exogenous parameters, and so on. In another version, an agent's "type" is specified as a probabilistic belief about the exogenous parameters and about the other agents' "types".

Each of these two versions of the universal type space has a weakness that the other version does not have.[1] Belief hierarchies do not easily fit into the standard framework of decision theory, where a single probability measure is used to represent an agent's uncertainty. This difficulty does not arise if an agent's "type" is represented by one probability measure. But then, the specification of an agent's "type" in terms of beliefs about the other agents' "types" involves an element of circularity.

The work of Mertens and Zamir (1985), Brandenburger and Dekel (1993), and Heifetz (1993) suggests that these weaknesses do not matter.[2] In particular, the circularity involved in defining "types" in terms of beliefs over (other agents') "types" is eliminated by showing that each belief hierarchy of an agent defines a unique probability measure over exogenous data and other agents' belief hierarchies and that this probability measure can be used to specify an agent's beliefs about the other agents' "types". The mapping from belief hierarchies to beliefs that is thus obtained is shown to be a homeomorphism, which suggests that the two notions of universal type space are actually equivalent.

However, this homeomorphism theorem depends on the topologies that are imposed on the different spaces.[3] Mertens and Zamir (1985) assume

[1] For an extensive discussion, see Heifetz and Samet (1998).

[2] Following the informal discussion in Harsanyi (1967/68), the notion of universal type space was formlized by Mertens and Zamir (1985). Their result was subsequently generalized by Brandenburger and Dekel (1993) and Heifetz (1993).

[3] For a criticism, see Heifetz and Samet (1998, 1999). Heifetz and Samet (1999) observe that, without a suitable topological structure, Kolmogorov's extension theorem cannot be used, so there may be coherent belief hierarchies that do not correspond to beliefs about the other agents' types. Given this criticism, Heifetz and Samet (1998) propose a purely measure theoretic formulation that focuses on beliefs, rather than coherent belief hierarchies.

that the space Θ of exogenous parameters is a compact metric space and use a straightforward induction argument to show that, for each k, the space of beliefs of order k with the topology of weak convergence of probability measures is also a compact metric space. Given this specification of the spaces of beliefs of different orders, they treat the space U_i of coherent belief hierarchies of agent i as a subset of the product of the spaces of beliefs of different orders. Kolmogorov's extension theorem then provides them with a one-to-one and onto mapping from U_i to the space $\mathcal{M}(\Theta \times U_{-i})$ of probability measures on the space of exogenous parameters and other agents' belief hierarchies. This mapping is a homeomorphism if the spaces of belief hierarchies of the different agents are endowed with the product topology and if $\mathcal{M}(\Theta \times U_{-i})$ is endowed with the topology of weak convergence of probability measures.[4]

In assessing the continuity properties of an agent's behaviour with respect to his "type", it thus makes no difference whether the agent's "type" is specified in terms of a hierarchy of beliefs or in terms of a single probability measure on the exogenous parameters and the other agents' belief hierarchies.Under standard assumptions, in either formulation, the dependence of behaviour on types exhibits what Dekel et al. (2006) call the *upper strategic convergence property*, i.e. upper hemi-continuity of behaviour correspondences.

However, the product topology on the space of belief hierarchies is too coarse to provide for what Dekel et al. (2006) call the *lower strategic convergence property*, namely the property that the minimal ε for which an agent's choices are strictly ε interim correlated rationalizable should depend continuously on the agent's belief hierarchy. The reason is that, under the product topology, belief hierarchies can be treated as similar even the associated beliefs of some order k are very different, provided that k is very large. The set of ε interim correlated rationalizable actions can be sensitive to differences in beliefs of arbitrarily high orders. Rubinstein's (1989) electronic mail game provides an example.

Dekel et al. (2006) and Chen et al. (2010, 2012) have therefore proposed finer topologies for the space of belief hierarchies. The *uniform strategic topology* of Dekel et al. (2006) is defined directly in terms of the desired upper and lower convergence properties of ε interim correlated rationaliz-

[4] Mertens and Zamir (1985) also showed that, if the space of belief hierarchies is endowed with the product topology, then every abstract (Harsanyi) type space can be mapped continuously into the space of belief hierarchies; if the abstract type space is nonredundant, the abstract type space is homeomorphic to a subspace of the space of belief hierarchies. For a discussion of the role of the chosen topologies in this result, see Section 7 below.

able choices, with a class of open sets specified so that these continuity properties hold uniformly over all strategic games with the given exogenous parameter spaces. The *uniform weak topology* of Chen et al. (2010, 2012) is derived from the topologies on the spaces of beliefs of different orders in the belief hierarchy, like the product topology. However, whereas, for very large k, beliefs of order k do not matter very much under the product topology, the uniform weak topology gives equal weight to all orders of beliefs. This property ensures that the uniform weak topology provides for both the lower and the upper strategic convergence properties of Dekel et al. (2006). Indeed, as Chen et al. (2010, 2012) show, the uniform weak topology coincides with the uniform strategic topology of Dekel et al. (2006). Given this coincidence, I propose to refer to this finer topology simply as the *uniform topology* on the universal type space.

The work of Dekel et al. (2006) and Chen et al. (2010, 2012) raises the question what the uniform topology on the space of belief hierarchies implies for the space of beliefs about exogenous parameter and other agents' belief hierarchies. Can we still say that each belief hierarchy of an agent defines a unique probability measure over exogenous data and other agents' belief hierarchies? Do we still get a homeomorphism between the spaces U_i and $\mathcal{M}(\Theta \times U_{-i})$?

This paper gives a positive answer to both these questions. I will show that, if the spaces of belief hierarchies of all agents are endowed with this topology and if the space $\Theta \times U_{-i}$ is endowed with the induced product topology, then any coherent belief hierarchy of agent i is consistent with is a unique probability measure on $\Theta \times U_{-i}$. Moreover, if the space of probability measures on $\Theta \times U_{-i}$ is endowed with the topology of weak convergence, the mapping from belief hierarchies to beliefs of agent i is a homeomorphism.

Several points should be noted. First, the imposition of the uniform rather than the product topology concerns the range as well as the domain of the mapping from U_i to $\mathcal{M}(\Theta \times U_{-i})$. This is important if the mapping from U_i to $\mathcal{M}(\Theta \times U_{-i})$ is to be a homeomorphism with *both*, the uniform and the product topology.

Second, if the spaces U_j of belief hierarchies of agents $j \neq i$ have the uniform topology, then Kolmogorov's extension theorem by itself is not enough to obtain a probability measure on $\Theta \times U_{-i}$ from a given belief hierarchy in U_i. The reason is that, as shown by Chen et al. (2016), the Borel σ-algebra on the space of belief hierarchies that is generated by the uniform topology is strictly larger than the product σ-algebra (or the Borel σ-algebra generated by the product topology).[5]

However, in Hellwig (2017), I show that, for any infinite product of complete separable metric spaces, any set in the Borel σ-algebra that is generated by the uniform topology is *universally measurable* in the sense of being measurable by the completion of any measure on the product σ-algebra.[6] The result implies that, even though the Borel σ-algebra that is induced by the uniform topology on the universal type space is larger than the product σ-algebra, the hierarchy of beliefs of different (finite) orders is still sufficient to pin down a unique probability measure on this larger σ-algebra. In fact, the measures on the Borel σ-algebra that is induced by the uniform topology are just the completions of the measures on the product σ-algebra, i.e. *all* probabililty measures on the larger σ-algebra can in fact be obtained from the agent's belief hierarchies.

Third, whereas Chen et al. (2010) have pointed out that the universal type space with the uniform topology is not separable, in Hellwig (2017), I show that any measure on the Borel σ-algebra that is induced by the uniform topology on a product of complete separable metric spaces has a separable support. This result ensures that the topology of weak convergence on the space of such measures is metrizable by the Prohorov metric. The proof of the homeomorphism theorem makes extensive use of this result.

The topology on the universal type space and the homeomorphism theorem matter not only for continuity but also for genericity considerations. An example is provided by the scope for full surplus extraction in mechanism design with correlated values, studied in Gizatulina and Hellwig (2017). The analysis in Gizatulina and Hellwig (2017) focuses on a condition that McAfee and Reny (1992) introduced as being necessary and sufficient for full surplus extraction to be feasible. This condition refers to beliefs as probability measures over the other agents' types. In the context of the universal type space, any such condition on beliefs about other agents' types raises questions about the implications of this condition for belief hierarchies. The homeomorphism theorem of this paper implies that, if the universal type space has the uniform topology, any genericity results that are obtained

[5] Whereas Chen et al. (2010, p. 459) claim that the two σ-algebras coincide, Chen et al. (2016) retract this claim and give an example of an open set in the universal weak topology that is not a Borel set of the product topology, namely the ε-neighbourhood of an analytic set that is not a Borel set.

[6] The completion of a measure μ that is defined on the product σ-algebra is a measure $\bar{\mu}$ that is defined on the smallest σ-algebra containing the product σ-algebra as well as the sets to which μ assigns outer measure zero and that assigns the same measure as μ to sets in the product σ-algebra and measure zero to sets to which μ assigns outer measure zero.

from the McAfee-Reny condition can easily be translated into genericity results about the space of belief hierarchies.

In the following, Section 2 introduces notation and some mathematical basics. Section 3 specifies the space of belief hierarchies and the different topologies on this space. Section 4 introduces the Kolmogorov mapping from the space of belief hierarchies to the space of beliefs about exogenous parameters and other agents' belief hierarchies when the latter space has the product topology. Section 5 shows that, when the universal type space has the uniform topology, the associated Borel sets are universally measurable and an extension of the Kolmogorov mapping can be used to specify measures for these sets. Section 6 states and proves the homeomorphism theorem for the universal type space with the uniform topology.

Section 7 concludes with a discussion of the relation between the "universal" type space and abstract (Harsanyi) type spaces when the former has the uniform topology; in this section, I show in particular, that the canonical mapping from abstract type spaces to the space of belief hierarchies with the uniform topology is continuous if belief functions in the abstract-type-space model are continuous when the topology on the ranges of these functions, i.e. the spaces of agents' probabilistic beliefs, is induced by the toral-variation metric.

2 Notation and Mathematical Preliminaries

Given any metric space S, I will write $\mathcal{B}(S)$ for the Borel σ-algebra on S and $\mathcal{M}(S)$ for the space of probability measures on $(S, \mathcal{B}(S))$. When it is appropriate, I will use superscripts to indicate which topology a given space S is endowed with. For example, if U_i is the space of agent i's belief hierarchies, I will write U_i^p and U_i^u to indicate whether U_i has the product topology or the uniform topology. Accordingly, I also distinguish between $\mathcal{M}(U_i^p)$ and $\mathcal{M}(U_i^u)$, the space of probability measures on $(U_i^p, \mathcal{B}(U_i^p))$ and the space of probability measures on $(U_i^u, \mathcal{B}(U_i^u))$.

Given a metric space S, I endow the space $\mathcal{M}(S)$ of probability measures on $(S, \mathcal{B}(S))$ with the topology of weak convergence of probability measures. In this topology, a sequence $\{\mu^r\}$ in $\mathcal{M}(S)$ converges to a limit μ if and only if, for every bounded and continuous real-valued function on S, the integrals $\int f(s)\mu^r(ds)$ converge to $\int f(s)\mu(ds)$.[7]

[7] As is well known, if S is a *separable* metric space, $\mathcal{M}(S)$ can be identified with the space of continuous linear functionals on the space $\mathcal{C}(S)$ of bounded continuous real-valued functions on S, i.e. the dual of $\mathcal{C}(S)$, and the topology of weak convergence coincides with

If all probability measures in $\mathcal{M}(S)$ have separable supports, the topology of weak convergence on $\mathcal{M}(S)$ can be metrized by the Prohorov metric, which specifies the distance $\rho(\mu, \hat{\mu})$ between two measures μ and $\hat{\mu}$ on S as the infimum of the set of ε such that

$$\mu(B) \leq \hat{\mu}(B^{\varepsilon}) + \varepsilon \text{ and } \hat{\mu}(B) \leq \mu(B^{\varepsilon}) + \varepsilon \tag{1}$$

for all sets $B \in \mathcal{B}(S)$ with ε-neighbourhoods $B^{\varepsilon} \in \mathcal{B}(S)$.[8] The condition that all probability measures in $\mathcal{M}(S)$ have separable supports is automaticaly satisfied if S itself is separable but is more general. As discussed in Appendix III, pp. 236 f., of Billingsley (1968), this condition is always satisfied if the cardinal of S is non-measurable, which in turn is true if the continuum hypothesis is assumed. In Hellwig (2017), I show that, without any presumption about the non-measurability of S, all probability measures in $\mathcal{M}(S)$ have separable supports if S is a product of complete separable metric spaces and S is endowed with the topology induced by the uniform metric.

The topology on S affects $\mathcal{M}(S)$ and the topology of weak convergence on $\mathcal{M}(S)$ in two ways. First, the topology on S determines the Borel σ-algebra $\mathcal{B}(S)$ and hence the domain of the measures in $\mathcal{M}(S)$. Given two topologies $\mathcal{T}^c, \mathcal{T}^f$ on S such that $\mathcal{T}^f$ is finer than $\mathcal{T}^c$, the Borel σ-algebra $\mathcal{B}(S^f)$ that is generated by $\mathcal{T}^f$ is no smaller than and in some cases strictly larger than the Borel σ-algebra $\mathcal{B}(S^c)$ that is generated by $\mathcal{T}^c$, where the notation S^c, S^f indicates whether S has the topology $\mathcal{T}^c$ or the topology $\mathcal{T}^f$.

Second, the topology on S determines the set of bounded and continuous real-valued functions on S. Given two topologies $\mathcal{T}^c, \mathcal{T}^f$ on S such that $\mathcal{T}^f$ is finer than $\mathcal{T}^c$, the set of bounded and continuous real-valued functions on S^f is larger than the set of bounded continuous functions on S^c. The requirement that integrals of all bounded continuous functions converge is more restrictive when S has the topology $\mathcal{T}^f$ than when S has the topology $\mathcal{T}^c$. Thus, the weak convergence of a sequence of measures in $\mathcal{M}(S^f)$ implies the weak convergence in $\mathcal{M}(S^c)$ of the restrictions of these measures to $\mathcal{B}(S^c)$, the weak convergence of a sequence of measures in $\mathcal{M}(S^c)$ does not generally imply the weak convergence in $\mathcal{M}(S^f)$ of the extensions of these measures to $\mathcal{B}(S^f)$. In the specification of the Prohorov metric, the

the weak* topology. If S is *not* separable, the dual of $\mathcal{C}(S)$ corresponds to the space $rba(S)$ of regular (finitely) additive set functions on $(S, B(S))$, which is larger than $\mathcal{M}(S)$. The topology of weak convergence on $\mathcal{M}(S)$ is then equivalent to the subspace topology that is induced by the weak* topology on $rba(S)$. See, e.g., Parthasarathy (1967), p. 35.

[8] See, e.g., Theorem 5, p. 238, and the discussion on p. 239 in Billingsley (1968).

dependence of the topology on $\mathcal{M}(S)$ on the topology on S is implicit in (i) the fact that the sets B for which (1) must hold depend on the topology and (ii) the fact that, for any B, the ε-neighbourhood $B^\varepsilon \in \mathcal{B}(S)$ that appears in (1) depends on the metric on S.

3 The Space of Belief Hierarchies

Following Chen et al. (2010), I use the procedure of Mertens and Zamir (1985) to construct the universal type space. Suppose that there are I agents. Let Θ be a compact metric space of exogenous factors that affect the strategic situation. Proceeding inductively, define a sequence of spaces $X^0, X^1, X^2, ...$ by setting

$$X^0 = \Theta, X^1 = X^0 \times \mathcal{M}(X^0)^{I-1} \tag{2}$$

and, for each $k \geq 2$,

$$X^k = \left\{ (\theta, \mu^1, ..., \mu^k) \in X^0 \times \prod_{\ell=1}^{k} \mathcal{M}(X^{\ell-1})^{I-1} : \operatorname{marg}_{X^{\ell-2}} \mu^\ell = \mu^{\ell-1}, \ell = 2, ..., k \right\}. \tag{3}$$

For any k, standard arguments imply that, if X^{k-1} is a compact metric space, then $\mathcal{M}(X^{k-1})$ and $X^0 \times \prod_{\ell=1}^{k} \mathcal{M}(X^{\ell-1})^{I-1}$ are also compact metric spaces and so is X^k, which is a closed subset of $X^0 \times \prod_{\ell=1}^{k} \mathcal{M}(X^{\ell-1})^{I-1}$. The assumption that $X^0 = \Theta$ is a compact metric space thus implies that, for every k, X^k and $\mathcal{M}(X^k)$ are compact metric spaces.[9]

The space of belief hierarchies of player i is defined as:

$$U_i = \left\{ (\mu^k)_{k \geq 1} \in \prod_{k \geq 1} \mathcal{M}(X^{k-1}) : \operatorname{marg}_{X^{k-2}} \mu^\ell = \mu^{\ell-1}, k = 2, 3, ... \right\}.^{10} \tag{4}$$

This space is obviously a subset of the product

$$\bar{U}_i := \prod_{k \geq 1} \mathcal{M}(X^{k-1}). \tag{5}$$

[9] Parthasarathy (1967), p. 45.

[10] Note that the right-hand side of (3) does not depend on i. The subscript i on the left-hand side is therefore irrelevant, a mere mnemonic device indicating that we are talking about the beliefs of agent i.

Given the topologies on $\mathcal{M}(X^{k-1})$ and the induced Borel σ-algebras $\mathcal{B}(\mathcal{M}(X^{k-1}))$, for $k = 1, 2, ...,$ let

$$\mathcal{B}(\bar{U}_i) := \prod_{k \geq 1} \mathcal{B}(\mathcal{M}(X^{k-1})) \tag{6}$$

be the corresponding product σ-algebra on $\bar{U}_i$ and

$$\mathcal{B}(U_i) := \{B \cap U_i | B \in \mathcal{B}(\bar{U}_i)\} \tag{7}$$

the induced σ-algebra on U_i.

The definition of the product σ-algebra is independent of the topology on U_i, but of course we have $\mathcal{B}(\bar{U}_i) = \mathcal{B}(\bar{U}_i^p)$ and

$$\mathcal{B}(U_i) = \mathcal{B}(U_i^p) \tag{8}$$

where the superscript π indicates that $\bar{U}_i$ and U_i are endowed with the product topology that is induced by the topologies on $\mathcal{M}(X^0)$, $\mathcal{M}(X^1), ...$ The product topology on $\bar{U}_i$ and U_i can be metrized, e.g., by the metric

$$\rho_i^p((\mu^k)_{k \geq 1}, (\hat{\mu}^k)_{k \geq 1}) := \sum_{k=1}^{\infty} \alpha^k \rho^k(\mu^k, \hat{\mu}^k), \tag{9}$$

where α is some number in the open interval $(0, 1)$ and, for any k and any two measures μ^k and $\hat{\mu}^k$ in $\mathcal{M}(X^{k-1})$, $\rho^k(\mu^k, \hat{\mu}^k)$ is the Prohorov distance between μ^k and $\hat{\mu}^k$, i.e.,

$$\begin{aligned}\rho^k(\mu^k, \hat{\mu}^k) = \inf\{\delta > 0 |\ \mu^k(B) \leq \hat{\mu}^k(B^\delta) + \delta \text{ and } \hat{\mu}^k(B) \leq \mu^k(B^\delta) + \delta \\ \text{for all } B \in \mathcal{B}(X^{k-1})\},\end{aligned} \tag{10}$$

where B^δ is the δ-neighbourhood of B in $X^{k-1} \subset X^0 \times \prod_{\ell=1}^{k-1} \mathcal{M}(X^{\ell-1})^{I-1}$.

As an alternative to the product topology, Chen et al. (2010) introduced what I propose to call the *uniform* topology. The uniform topology on U_i is induced by the metric ρ^u such that

$$\rho^u((\mu^k)_{k \geq 1}, (\hat{\mu}^k)_{k \geq 1}) := \sup_k \rho^k(\mu^k, \hat{\mu}^k) \tag{11}$$

for any $(\mu^k)_{k \geq 1}$ and $(\hat{\mu}^k)_{k \geq 1}$ in U_i. I will use the notation U_i^u to indicate that U_i is endowed with the uniform topology. The uniform topology on U_i

is obviously finer than the product topology. In fact, whereas U_i^p, a closed subset of the product $\bar{U}_i$ of compact metric spaces, is itself a compact metric space, Chen et al. (2010) show that U_i^u is not even separable. Chen et al. (2016) also show that the Borel σ-algebra $\mathcal{B}(U_i^u)$ that is induced by the uniform topology on U_i is strictly larger than $\mathcal{B}(U_i^p)$ and $\mathcal{B}(U_i)$.

4 From Belief Hierarchies to Beliefs: The Kolmogorov Mapping

I now turn to the relation between the space U_i of belief hierarchies for agent i and the space of probability measures on the product $\Theta \times U_{-i}$, where $U_{-i} := \prod_{j \neq i} U_j$ is the space of vectors of belief hierarchies of the other agents. Along the lines of the preceding discussion, for $j \neq i$, let $\mathcal{B}(U_j)$ be the product σ-algebra on $\prod_{k \geq 1} \mathcal{B}(\mathcal{M}((X^k))$, let $\mathcal{B}(U_{-i}) = \prod_{j \neq i} \mathcal{B}^\infty(U_j)$, and let $\mathcal{M}(\Theta \times U_{-i})$ be the set of probability measures on $(\Theta \times U_{-i}, \mathcal{B}(\Theta) \times \mathcal{B}(U_{-i}))$.

Mertens and Zamir (1985) noted that, for every belief hierarchy $(\mu^k)_{k \geq 1} \in U_i$ of agent i, there exists a unique probability measure

$$\mu^\infty = \beta_i((\mu^k)_{k \geq 1}) \in \mathcal{M}(\Theta \times U_{-i}) \tag{12}$$

such that the marginal distributions on $X^1, X^2, ...$ that are induced by $\mu^\infty = \beta_i((\mu^k)_{k \geq 1})$ are just the measures μ^k, $k = 1, 2, ...$ Thus, for any k and any $B_\Theta \in \mathcal{B}(\Theta)$ and $B_j^\ell \in \mathcal{B}(\mathcal{M}(X^{\ell-1}))$, $j \neq i, \ell = 1, ..., k$,

$$\begin{aligned}\mu^\infty \left(B_\Theta \times \prod_{j \neq i} [B_j^1 \times ... \times B_j^k \times \mathcal{M}(X^k) \times \mathcal{M}(X^{k+1}) \times ...] \right) \qquad (13) \\ = \mu^k \left(B_\Theta \times \prod_{j \neq i} B_j^1 \times ... \times \prod_{j \neq i} B_j^k \right).\end{aligned}$$

The consistency condition $\mathrm{marg}_{X^{k-2}} \mu^\ell = \mu^{\ell-1}, \ell = 1, 2, ...$, ensures that, for any $k' > k$, we have

$$\begin{aligned}\mu^{k'} \left(B_\Theta \times \prod_{j \neq i} B_j^1 \times ... \times \prod_{j \neq i} B_j^k \times \mathcal{M}(X^k)^{I-1} \times ... \times \mathcal{M}(X^{k'-1})^{I-1} \right) \\ = \mu^k \left(B_\Theta \times \prod_{j \neq i} B_j^1 \times ... \times \prod_{j \neq i} B_j^k \right), \qquad (14)\end{aligned}$$

so the value of $\mu^\infty\left(B_\Theta \times \prod_{j\neq i}[B_j^1 \times ... \times B_j^k \times \mathcal{M}(X^k) \times \mathcal{M}(X^{k+1}) \times ...]\right)$ is independent of whether we use (13) with the given k, B_Θ, and B_j^ℓ, $j \neq i, \ell = 1,...,k$, or whether we use (13) with $k' > k, k$, B_Θ, and B_j^ℓ, $j \neq i, \ell = 1,...,k'$, with $B_j^\ell = \mathcal{M}(X^\ell)$ for $\ell \in \{k,...,k'\}$. The set function that is given by condition (13) defines a finitely additive meassure on the algebra of products of the form $B_\Theta \times \prod_{j\neq i}[B_j^1 \times ... \times B_j^k \times \mathcal{M}(X^k) \times \mathcal{M}(X^{k+1}) \times ...]$, $k = 1, 2, ...$, By Kolmogorov's extension theorem, this set function can be uniquely extended to a countably additive measure μ^∞ on $\mathcal{B}(\Theta) \times \mathcal{B}^\infty(U_{-i})$.[11]

I will refer to the mapping $(\mu^k)_{k\geq1} \rightarrow \beta_i((\mu^k)_{k\geq1})$ that is thus obtained as the *Kolmogorov mapping.* The Kolmogorov mapping is one-to-one and onto, i.e., for any probability measure $\mu^\infty \in \mathcal{M}^\infty(\Theta \times U_{-i})$, there exists a unique belief hierarchy $(\mu^k)_{k\geq1} \in U_i$ such that $\mu^\infty = \beta_i((\mu^k)_{k\geq1})$. To see this, it suffices to note that, for any $\mu^\infty \in \mathcal{M}^\infty(\Theta \times U_{-i})$ and any k, (13) defines a measure μ^k on the algebra of products of the form $B_\Theta \times \prod_{j\neq i} B_j^1 \times ... \times \prod_{j\neq i} B_j^k$ and that this measure can be uniquely extended to a measure on $(X^k, \mathcal{B}(X^k))$.[12]

If the spaces of belief hierarchies are given the product topology, we obviously have $\mathcal{B}(U_j^p) = \mathcal{B}(U_j)$ for all j and therefore $\mathcal{B}(\Theta \times U_{-i}^p) = \mathcal{B}(\Theta) \times \mathcal{B}(U_{-i})$, where

$$\mathcal{B}(U_{-i}^p) := \prod_{j\neq i} \mathcal{B}(U_j^p) \tag{15}$$

is the Borel σ-algebra on the product $\prod_{j\neq i} U_j$ when the spaces U_j have the product topology. The sets $\mathcal{M}(\Theta \times U_{-i})$ and $\mathcal{M}(\Theta \times U_{-i}^p)$ are the same, and we may think of the Kolmogorov mapping as a mapping from U^p to $\mathcal{M}(\Theta \times U_{-i}^p)$. As was shown by Mertens and Zamir (1985), this mapping is a homeomorphism.

[11] Billingsley (1968), p. 228, Dudley (2002), p. 257.

[12] Halmos (1950), p. 54, Dudley (2002), pp. 89ff.

5 The Extended Kolmogorov Mapping

If the spaces of belief hierarchies are given the uniform topology, matters are more complicated. As mentioned above, the σ-algebra $\mathcal{B}(U_i^u)$ is strictly larger than the product σ-algebra $\mathcal{B}^\infty(U_i)$). This finding of Chen et al. (2016) raises the question whether the hierarchies of beliefs of different orders contain enough information to pin down an agent's belief as a measure on $(\Theta \times U_{-i}^u, \mathcal{B}(\Theta \times U_{-i}^u))$.

The answer to this question turns out to be positive. The following result shows that the sets in $\mathcal{B}(\Theta \times U_{-i}^u)$ are in fact *universally measurable* in the sense that, for every measure $\mu \in \mathcal{M}(\Theta \times U_{-i}^p)$, they belong to the completion $\bar{\mathcal{B}}(\Theta \times U_{-i}^p, \mu)$ of $\mathcal{B}(\Theta \times U_{-i}^p)$ that is determined by μ. The completion of the σ-algebra $\mathcal{B}(\Theta \times U_{-i}^p)$ that is determined by μ is the σ-algebra that is generated by $\mathcal{B}(\Theta \times U_{-i}^p)$ and the class $\mathcal{N}(\mu)$ of subsets of $\Theta \times U_{-i}^p$ that are contained in sets $B \in \mathcal{B}(\Theta \times U_{-i}^p)$ for which $\mu(B) = 0$.

Proposition 1 $\mathcal{B}(\Theta \times U_{-i}^u) \subset \bar{\mathcal{B}}(\Theta \times U_{-i}^p, \mu)$ *for all* $\mu \in \mathcal{M}(\Theta \times U_{-i}^p)$.

Proof. Consider the product

$$Y := \Theta \times \mathcal{M}(X^0)^{I-1} \times \mathcal{M}(X^1)^{I-1} \times .... \tag{16}$$

In this product, each of the factor spaces Θ, $\mathcal{M}(X^0)^{I-1}$, $\mathcal{M}(X^1)^{I-1}, ...$ is a compact metric space. Let $d_\Theta, d_0^Y, d_1^Y, ...$ be the metrics on these factor spaces, and define a uniform metric d^u so that, for any y and $\hat{y}$ in Y, $d^u(y, \hat{y})$ is the supremum of the distances between the projections of y and $\hat{y}$ to Θ, $\mathcal{M}(X^0)^{I-1}$, $\mathcal{M}(X^1)^{I-1}, ...$ Let $\mathcal{B}^u(Y)$ be the Borel σ-algebra that is induced by the associated uniform-metric topology on Y. Further, let $\mathcal{B}^p(Y)$ be the Borel product σ-algebra on Y and, for any measure μ^Y on $(Y, \mathcal{B}^p(Y))$, let $\bar{\mathcal{B}}^p(Y, \mu^Y)$ be the completion of $\mathcal{B}^p(Y)$ that is determined by μ^Y, i.e. the smallest σ-algebra that contains $\mathcal{B}^p(Y)$ as well as all sets that are contained in sets to which μ^Y assigns the measure zero. By Proposition 1 in Hellwig (2017), $\mathcal{B}^u(Y) \subset \bar{\mathcal{B}}^p(Y, \mu^Y)$ for all measures μ^Y on $(Y, \mathcal{B}^p(Y))$.

Proposition 1 follows by observing that the mapping

$$(\theta, (\mu_j^k)_{k\geq 1, j\neq i}) \to f(\theta, (\mu_j^k)_{k\geq 1, j\neq i}) = (\theta, (\mu_j^1)_{j\neq i}, (\mu_j^2)_{j\neq i}, ...), \tag{17}$$

which simply reorders the terms in the sequence $(\mu_j^k)_{k\geq 1, j\neq i}$, is a Borel isomorphism between $(\Theta \times U_{-i}^u, \mathcal{B}(\Theta \times U_{-i}^u))$ and $(Y, \mathcal{B}^u(Y))$ and also a Borel isomorphism between $(\Theta \times U_{-i}^p, \mathcal{B}(\Theta \times U_{-i}^p))$ and $(Y, \mathcal{B}^p(Y))$. ■

Corollary 2 *Every probability measure $\mu \in \mathcal{M}(\Theta \times U^p_{-i})$ can be uniquely extended to a measure $\bar{\mu}(\mu) \in \mathcal{M}(\Theta \times U^u_{-i})$. The mapping $\mu \to \bar{\mu}(\mu)$ from $\mathcal{M}(\Theta \times U^p_{-i})$ to $\mathcal{M}(\Theta \times U^u_{-i})$ is one-to-one and onto.*

Proof. For any measure $\mu \in \mathcal{M}(\Theta \times U^p_{-i})$, let μ^* be the outer measure that μ induces on the subsets of $\Theta \times U_{-i}$, and let $\mathcal{N}(\mu)$ be the class of subsets of $\Theta \times U_{-i}$that have outer measure zero. Then $\bar{\mathcal{B}}(\Theta \times U^p_{-i}, \mu)$ is the σ-algebra that is generated by $\mathcal{B}^p(Y) \cup \mathcal{N}(\mu)$. As discussed in Section 3.3 of Dudley (2002), by setting $\bar{\mu}(B) = \mu^*(B)$ for $B \in \bar{\mathcal{B}}(\Theta \times U^p_{-i}, \mu)$, one obtains an extension of μ to a measure $\bar{\mu}(\mu)$ on $\bar{\mathcal{B}}(\Theta \times U^p_{-i}, \mu)$. By Proposition 1, $\mathcal{B}(\Theta \times U^u_{-i})$ is contained in $\bar{\mathcal{B}}(\Theta \times U^p_{-i}, \mu)$, and the restriction of $\bar{\mu}(\mu)$ to $\mathcal{B}(\Theta \times U^u_{-i})$ is a probability measure on $(\Theta \times U^u_{-i}, \mathcal{B}(\Theta \times U^u_{-i}))$ that agrees with μ on $\mathcal{B}(\Theta \times U^p_{-i})$. Uniqueness follows from the fact that, by the monotonicity of measures, for each set $W \in \mathcal{N}(\mu)$, the measure assigned to W must be zero. This proves the first statement of the corollary.

Since $\Theta \times U_{-i} \in \mathcal{B}(\Theta \times U^p_{-i}) \subset \mathcal{B}(\Theta \times U_{-i})$, for every probability measure $\bar{\mu} \in \mathcal{M}(\Theta \times U^u_{-i})$, the restriction μ of $\bar{\mu}$ to $\mathcal{B}(\Theta \times U^p_{-i})$ is a probability measure, i.e. an element of $\mathcal{M}(\Theta \times U^p_{-i})$. The second statement of the corollary follows immediately. ■

The composition of the mapping $\mu \to \bar{\mu}(\mu)$ in Corollary 2 with the Kolmogorov mapping is a mapping

$$\bar{\beta}_i := \bar{\mu} \circ \beta_i$$

from the space U_i of agent i's belief hierarchies to the space $\mathcal{M}(\Theta \times U^u_{-i})$ of probability measures on $(\Theta \times U^u_{-i}, \mathcal{B}(\Theta \times U^u_{-i}))$. I will refer to this mapping as the *extended Kolmogorov mapping.* The extended Kolmogorov mapping plays the same role for the universal type space with the uniform topology as the Kolmogorov mapping itself plays for the universal type space with the product topology.

In the analysis of the extended Kolmogorov mapping below, I will make use of the following additional result.

Proposition 3 *The topology of weak convergence on $\mathcal{M}(\Theta \times U^u_{-i})$ is metrizable by the Prohorov metric.*

Proposition 3 follows from Proposition 3 and Corollary 4 in Hellwig (2017). Proposition 3 in Hellwig (2017) states that, for a product of the form (16), every measure in $\mathcal{M}(Y^u)$ has a separable support. Given that the mapping (17) is a Borel isomorphism, it follows that every measure in

$\mathcal{M}(\Theta \times U^u_{-i})$ has a separable support. Metrizability of the topology of weak convergence by the Prohorov metric then follows from Theorem 5, p. 238, in Billingsley (1968).

6 The Homeomorphism Theorem for the Extended Kolmogorov Mapping

For the extended Kolmogorov mapping, the following result provides an analogue to the homeomorphism theorem of Mertens and Zamir (1985).

Proposition 4 *Assume that $\mathcal{M}(\Theta \times U^u_{-i})$ has the topology of weak convergense. Then the extended Kolmogorov mapping defines a homeomorphism between U^u_i and $\mathcal{M}(\Theta \times U^u_i)$.*

Proof. Because the mapping $\bar{\beta}_i$ is injective and onto, it suffices to show that, both $\bar{\beta}_i$ and its inverse $\bar{\beta}_i^{-1}$ are continuous.

I first show that $\bar{\beta}_i$ is continuous. Proceeding indirectly, suppose that $\bar{\beta}_i$ is *not* continuous. Then there exists a sequence $\left\{(\mu^{kr})_{k\geq 1}\right\}_{r=1}^{\infty}$ and there exists $(\mu^k)_{k\geq 1}$ such that $(\mu^{kr})_{k\geq 1} \in U_i$ for all r, $(\mu^k)_{k\geq 1} \in U_i$,

$$\lim_{r\to\infty} \rho^u_i((\mu^{kr})_{k\geq 1}, (\mu^k)_{k\geq 1}) = 0, \tag{18}$$

but $\bar{\beta}_i((\mu^{kr})_{k\geq 1})$ does not converge to $\bar{\beta}_i((\mu^k)_{k\geq 1})$. To simplify the notaton, we write $u^r = (\mu^{kr})_{k\geq 1}$, $r = 1, 2, ...$, and $u = (\mu^k)_{k\geq 1}$. Taking subsequences if necessary, we may suppose that, for some $\varepsilon > 0$, the Prohorov distance between $\bar{\beta}_i(u^r)$ and $\bar{\beta}_i(u)$ exceeds ε for all r. For each r, therefore, there exists a set $W^r \in \mathcal{B}(\Theta \times U^u_{-i})$, with ε-neighbourhood $(W^r)^\varepsilon$, such that either

$$\bar{\beta}_i(W^r|u^r) > \bar{\beta}_i((W^r)^\varepsilon|u) + \varepsilon \tag{19}$$

or

$$\bar{\beta}_i(W^r|u) > \bar{\beta}_i((W^r)^\varepsilon|u^r) + \varepsilon. \tag{20}$$

For any n and any $j \neq i$, let U^n_j by the projection of U_j to the space $\mathcal{M}(X^0) \times ... \times \mathcal{M}(X^n)$, and let $U^n_{-i} := \prod_{j\neq i} U^n_j$. Further, let W^{nr} be the projection of W^r to $\Theta \times U^n_{-i}$ and let $(W^{nr})^\varepsilon$ be an ε-neighbourhood (in $\Theta \times U^n_{-i}$) of W^{nr}. Let

$$\hat{W}^{nr} = W^{nr} \times \prod_{j\neq i} [\mathcal{M}(X^n_{-j}) \times \mathcal{M}(X^{n+1}_{-j}) \times ...] \tag{21}$$

and

$$\hat{W}^{nr\varepsilon} = (W^{nr})^{\varepsilon} \times \prod_{j\neq i}[\mathcal{M}(X^{n}_{-j}) \times \mathcal{M}(X^{n+1}_{-j}) \times ...] \tag{22}$$

be the cylinder sets in $\Theta \times U_{-i}$ that are defined by W^{nr} and $(W^{nr})^{\varepsilon}$. One easily verifies that the sequences $\{\hat{W}^{nr}\}_{n=1}^{\infty}$ and $\{\hat{W}^{nr\varepsilon}\}_{n=1}^{\infty}$ are nonincreasing and that

$$W^{r} = \bigcap_{n=1}^{\infty} \hat{W}^{nr} \text{ and } (W^{r})^{\varepsilon} = \bigcap_{n=1}^{\infty} \hat{W}^{nr\varepsilon} \tag{23}$$

for all r. By elementary measure theory,[13] it follows that, for any r and any $\delta > 0$, there exists $N^{r}(\delta)$ such that for $n > N^{r}(\delta)$,

$$\bar{\beta}_{i}((W^{r})^{\varepsilon}|u) \geq \bar{\beta}_{i}(\hat{W}^{nr\varepsilon}|u) - \delta \tag{24}$$

and

$$\bar{\beta}_{i}((W^{r})^{\varepsilon}|u^{r}) \geq \bar{\beta}_{i}(\hat{W}^{nr\varepsilon}|u^{r}) - \delta. \tag{25}$$

Moreover,

$$\bar{\beta}_{i}(W^{r}|u^{r}) \leq \bar{\beta}_{i}(\hat{W}^{nr}|u^{r}) \tag{26}$$

and

$$\bar{\beta}_{i}(W^{r}|u) \leq \bar{\beta}_{i}(\hat{W}^{nr}|u) \tag{27}$$

Upon combining (24) - (27) with (19) and (20), we find that, for all r and $\delta > 0$, there exists $N^{r}(\delta)$ such that for $n > N^{r}(\delta)$, either

$$\bar{\beta}_{i}(\hat{W}^{nr}|u^{r}) > \bar{\beta}_{i}(\hat{W}^{nr\varepsilon}|u) - \delta + \varepsilon \tag{28}$$

or

$$\bar{\beta}_{i}(\hat{W}^{nr}|u) > \bar{\beta}_{i}(\hat{W}^{nr\varepsilon}|u^{r}) - \delta + \varepsilon. \tag{29}$$

By (13) and the definitions of $u^{r} = (\mu^{kr})_{k\geq 1}$ and $u = (\mu^{k})_{k\geq 1}$, we also have

$$\bar{\beta}_{i}(\hat{W}^{nr}|u^{r}) = \mu^{nr}(W^{nr}), \bar{\beta}_{i}(\hat{W}^{nr\varepsilon}|u^{r}) = \mu^{nr}((W^{nr})^{\varepsilon}) \tag{30}$$

and

$$\bar{\beta}_{i}(\hat{W}^{nr}|u) = \mu^{n}(W^{nr}), \bar{\beta}_{i}(\hat{W}^{nr\varepsilon}|u) = \mu^{n}((W^{nr})^{\varepsilon}) \tag{31}$$

for all r and n. For any r and and any sufficiently large n. therefore, either

$$\mu^{n}(W^{nr}) > \mu^{nr}((W^{nr})^{\varepsilon}) - \delta + \varepsilon \tag{32}$$

or

$$\mu^{nr}(W^{nr}) > \mu^{n}((W^{nr})^{\varepsilon}) - \delta + \varepsilon. \tag{33a}$$

[13] See Halmos (1950), p.38.

If $\delta < \varepsilon$, we also have $(W^{nr})^{\varepsilon-\delta} \subset (W^{nr})^{\varepsilon}$ and $(W^{r})^{\varepsilon-\delta} \subset (W^{r})\varepsilon$, so one may infer that, for all r and $\delta \in (0, \varepsilon)$, there exists $N^{r}(\delta)$ such that for any $n > N^{r}(\delta)$, either

$$\mu^{n}(W^{nr}) > \mu^{nr}((W^{nr})^{\varepsilon-\delta}) - \delta + \varepsilon \tag{34}$$

or

$$\mu^{nr}(W^{nr}) > \mu^{n}((W^{nr})^{\varepsilon-\delta}) - \delta + \varepsilon. \tag{35a}$$

But then, for any r, for $n > N^{k}(\delta)$, the Prohorov distance between the measures μ^{nr} and μ^{n} is at least $\varepsilon - \delta$.

It follows that for all r, we have

$$\rho^{u}((\mu^{kr})_{k\geq 1}, (\mu^{k})_{k\geq 1}) \geq \varepsilon - \delta,$$

i.e. the distance between $u^{r} = (\mu^{kr})_{k\geq 1}$ and $u = (\mu^{k})_{k\geq 1}$ in the metric for the uniform topology is at least $\varepsilon - \delta > 0$. This conclusion contradicts the assumption that the sequence $u^{r} = (\mu^{kr})_{k\geq 1}$ converges to $u = (\mu^{k})_{k\geq 1}$ in the uniform topology. The assumption that the map $u_{i} \rightarrow \bar{\beta}_{i}(u_{i})$ is not continuous has thus led to a contradiction and must be false.

Continuity of the map $\bar{\beta}^{-1}$ from $\mathcal{M}(\Theta \times U_{-i}^{u})$ to U_{i}^{u} is easily obtained by observing that, for any set $W^{n} \in \mathcal{B}(X_{i}^{n})$ the associated cylinder set

$$\hat{W}^{n} = W^{n} \times \prod_{j\neq i}[\mathcal{M}(X_{-j}^{n}) \times \mathcal{M}(X_{-j}^{n+1}) \times ...] \tag{36}$$

belongs to $\mathcal{B}(\Theta \times U_{-i}^{u})$, and, for any $\varepsilon > 0$, the cylinder set

$$\hat{W}^{n\varepsilon} = (W^{n})^{\varepsilon} \times \prod_{j\neq i}[\mathcal{M}(X_{-j}^{n}) \times \mathcal{M}(X_{-j}^{n+1}) \times ...] \tag{37a}$$

that is defined by the ε-neighbourhood $(W^{n})^{\varepsilon}$ of W^{n} in X_{-i}^{n} is actually an ε-neighbourhood of $\hat{W}^{n}$ in $\Theta \times U_{-i}^{u}$. Hence, if the Prohorov distance between two measures μ^{∞} and $\hat{\mu}^{\infty}$ in $\mathcal{M}(\Theta \times U_{-i}^{u})$ is less than ε, we must have

$$\mu^{\infty}(\hat{W}^{n}) < \hat{\mu}^{\infty}(\hat{W}^{n\varepsilon}) + \varepsilon \tag{38}$$

and

$$\hat{\mu}^{\infty}(\hat{W}^{n}) < \mu^{\infty}(\hat{W}^{n\varepsilon}) + \varepsilon. \tag{39}$$

By the definition of the marginal distributions, it follows that

$$\mu^{n}(W^{n}) < \hat{\mu}^{n}(W^{n\varepsilon}) + \varepsilon \tag{40}$$

and

$$\hat{\mu}^n(W^n) < \mu^n(W^{n\varepsilon}) + \varepsilon. \tag{41}$$

Since the choice of $W^n \in \mathcal{B}(X_{-i}^n)$ was arbitrary, it follows that the Prohorov distance between μ^n and $\hat{\mu}^n$ is no greater than ε. Since ε may be taken to be arbitrarily close to the Prohorov distance between μ^∞ and $\hat{\mu}^\infty$, it follows that the Prohorov distance between μ^n and $\hat{\mu}^n$ is no greater than the Prohorov distance between μ^∞ and $\hat{\mu}^\infty$. Since this latter statement holds for all n, it follows that the supremum of the Prohorov distances between the marginal distributions μ^n and $\hat{\mu}^n$ for $n = 1, 2, ...$, is no greater than the Prohorov distance between μ^∞ and $\hat{\mu}^\infty$. Continuity of the map from measures on $\Theta \times U_{-i}^u$ to belief hierarchies in U_i^u follows immediately. ■

7 Abstract Types Spaces and the Universal Type Space with the Uniform Topology

The word "universal" reflects the notion that the universal type space has room for a complete representation of *all* strategically relevant aspects of a given specification of information incompleteness. Thus, Mertens and Zamir (1985) showed that every abstract (Harsanyi) type space with continuous belief functions can be mapped into the space of belief hierarchies. If the space of belief hierarchies is endowed with the product topology, this mapping is continuous; if the abstract type space is nonredundant, the mapping is actually an embedding, i.e. the abstract type space is homeomorphic to a subspace ot the space of belief hierarchies.

Like the homeomorphism theorem for U_i and $\mathcal{M}(\Theta \times U_{-i})$, this result depends on the chosen topologies. To conclude this paper, I study what becomes of the Mertens-Zamir result when the space U_i^u of belief hierarchies with the uniform topology?[14]

As before, let Θ be a compact metric space of exogenous factors. A Θ-based abstract-type-space model with I agents involves, for each agent i, a compact metric space T_i of "types" and a mapping $b_i : T_i \to \mathcal{M}(\Theta \times T_{-i})$ that indicates how the agent's beliefs about the exogenous factors and about the other agents' "types" depend on the agent's own type.

Given the Θ-based abstract-type-space model $\{T_i, b_i\}_{i=1}^I$, for any i and any $t_i \in T_i$, one obtains a unique belief hierarchy $\varphi_i^\infty(t_i) \in U_i$ by setting:

$$\varphi_i^\infty(t_i) = (\varphi_i^1(t_i), \varphi_i^2(t_i), ...), \tag{42}$$

[14] The importance of this question is extensively discussed by Morris (2002).

where

$$\varphi_i^1(t_i) = b_i(\cdot\,|t_i) \circ \left(\pi_\theta^i\right)^{-1}, \tag{43}$$

and, for $k = 2, 3, ...,$

$$\varphi_i^k(t_i) = b_i(\cdot\,|t_i) \circ \left(\pi_\theta^i \times \prod_{j \neq i} (\varphi_j^{k-1} \circ \pi_j^i)\right)^{-1}; \tag{44}$$

in (43) and (44), π_θ^i and π_j^i, $j \neq i$, denote the projections from $\Theta \times T_{-i}$ to Θ to T_j. To see that, for any i, φ_i^∞ maps T_i into U_i, it suffices to observe that, for any i and k, the function

$$(\theta, t_{-i}) \to \left(\pi_\theta^i \times \prod_{j \neq i} (\varphi_j^{k-1} \circ \pi_j^i)\right)(\theta, t_{-i}) = \left(\theta \times \prod_{j \neq i} (\varphi_j^{k-1}(t_j)\right)$$

takes values in the space X^{k-1} that is defined by (2) and (3), therefore the function φ_i^k takes values in $\mathcal{M}(X^{k-1})$.

One easily verifies that, if, for any i, the range $\mathcal{M}(\Theta \times T_{-i})$ of the belief mapping b_i is endowed with the topology of weak convergence and the belief mapping b_i is continuous, then, for any i and k, the mapping φ_i^k from T_i to $\mathcal{M}(X^{k-1})$ is continuous. If U_i is endowed with the product topology, it follows that the map φ_i^∞ from T_i to U_i is also continuous; this is the result of Mertens and Zamir (1985). The following assumption and proposition provide an analogue for the case where U_i is endowed with the uniform topology.

Condition 5 *For any i, the belief function $b_i : T_i \to \mathcal{M}(\Theta \times T_{-i})$ is continuous when $\mathcal{M}(\Theta \times T_{-i})$ has the topology induced by the total-variation metric*

$$\rho^{TV}(\nu, \hat{\nu}) = \sup_{B \in \mathcal{B}(\Theta \times T_{-i})} |\nu(B) - \hat{\nu}(B)|\,. \tag{45}$$

Proposition 6 *Under Condition 5, the mapping φ_i^∞ from T_i to U_i^u is continuous.*

Proof. For $k = 2, 3, ...,$ (44) implies that

$$\varphi_i^k(B^{k-1}|t_i) = b_i\left(\left(\pi_\theta^i \times \prod_{j \neq i} (\varphi_j^{k-1} \circ \pi_j^i)\right)^{-1} (B^{k-1})\,|t_i\right) \tag{46}$$

for any measurable set $B^{k-1} \in \mathcal{B}(X^{k-1})$ and all $t_i \in T_i$. By the definition of the Prohorov metric,

$$\rho^k(\varphi_i^k(t_i), \varphi_i^k(t_i')) \leq \sup_{B^{k-1} \in \mathcal{B}(X^{k-1})} \left| \varphi_i^k(B^{k-1}|t_i) - \varphi_i^k(B^{k-1}|t_i') \right| \tag{47}$$

for all t_i and t_i' in T_i. By (46), it follows that

$$\rho^k(\varphi_i^k(t_i), \varphi_i^k(t_i')) \leq \sup_{B \in \mathcal{B}(\Theta \times T_{-i})} \left| b_i(B|t_i) - b_i(B|t_i') \right| = \rho^{TV}(b_i(t_i), b_i(t_i')) \tag{48}$$

for all t_i and t_i' in T_i. By the same argument, one also has

$$\rho^1(\varphi_i^1(t_i), \varphi_i^1(t_i')) \leq \rho^{TV}(b_i(t_i), b_i(t_i')) \tag{49}$$

for all t_i and t_i' in T_i. Hence

$$\rho^u(\varphi_i^\infty(t_i), \varphi_i^\infty(t_i')) = \sup_k \rho^k(\varphi_i^k(t_i), \varphi_i^k(t_i')) \leq \rho^{TV}(b_i(t_i), b_i(t_i')) \tag{50}$$

for all t_i and t_i' in T_i. The proposition follows immediately. ∎

Refer to the abstract-type-space model $\{T_i, b_i\}_{i=1}^I$ as *nonredundant* if, for each i, the map $\varphi_i^\infty : T_i \to U_i$ is injective, so $t_i \neq t_i'$ implies $\varphi_i^\infty(t_i) \neq \varphi_i^\infty(t_i')$.

Corollary 7 *If the abstract-type-space model $\{T_i, b_i\}_{i=1}^I$ is nonredundant, then,under Condition 5, T_i is homeomorphic to the subset $\varphi_i^\infty(T_i)$ of U_i^u.*

To understand the difference between this result and the result of Mertens and Zamir (1985), consider the following version of Rubinstein's (1989) electronic mail game. Let $\Theta = \{0, 1\}$ and $I = 2$. For $i = 1, 2$, let $T_i = \{0, \frac{1}{2}, \frac{2}{3}, ..., 1\}$ and assume that T_i has the subspace topology that is induced by the usual topology on $[0, 1]$. Specify the belief function for agent 1 so that

$$b_1(0) = \delta_{(0,0)}, b_1(1) = \delta_{(1,1)}, \tag{51}$$

and, for $t_1 = \frac{n}{n+1} > 0$,

$$b_1(t_1) = \frac{1}{2}\delta_{(1,\frac{n-1}{n})} + \frac{1}{2}\delta_{(1,\frac{n}{n+1})}, \tag{52}$$

where for any $(\theta, t_2) \in \Theta \times T_2$, $\delta_{(\theta,t_2)}$ is the degenerate measure that assigns all probability mass to the singleton $\{(\theta, t_2)\}$. Similarly, specify the belief function for agent 2 so that

$$b_2(0) = \frac{1}{2}\delta_{(0,0)} + \frac{1}{2}\delta_{(1,\frac{1}{2})}, b_2(1) = \delta_{(1,1)}. \tag{53}$$

and, for $t_2 = \frac{n}{n+1} > 0$,

$$b_2(t_2) = \frac{1}{2}\delta_{(1, \frac{n}{n+1})} + \frac{1}{2}\delta_{(1, \frac{n+1}{n+2})}. \tag{54}$$

On easily verifies that these belief functions are continuous if the spaces $\mathcal{M}(\Theta \times T_{-i})$, $i = 1, 2$, are endowed with the topology of weak convergence and that they are discontinuous at $t_i = 1$ if the spaces $\mathcal{M}(\Theta \times T_{-i})$, $i = 1, 2$, are endowed with the topology induced by the total-variation metric. Moreover, the mappings φ_i^∞ from T_i to U_i are continuous if U_i has the product topology and discontinuous at $t_i = 1$ if U_i has the uniform topology. Because of the discontinuity, the abstract type space model fails to exhibit what Dekel et al. (2006) refer to as the lower strategic convergence property.

To see this, consider a strategic game in which each agent has a choice between two actions, a_0 and a_1. Suppose that, for each agent, action a_0 always gives the payoff zero, but action a_1 gives the payoff $Y > 0$ if $\theta = 1$ and if the other agent also chooses the action a_1 and the payoff $-X < 0$ otherwise, where $X > Y$. Then, one easily verifies that, for each agent i, if $\varepsilon < \frac{1}{2}(Y - X)$, then for all $t_i \in T_i \backslash \{1\}$, a_0 is the unique ε interim correlated rationalizable action of agent i with the abstract type t_i, but for $t_i = 1$, the action a_1 is interim rationalizable. The lower strategic convergence property of Dekel et al. (2006) fails to hold.

This failure of the lower strategic convergence property also implies that, in the given example, the mappings φ_i^∞ from T_i into U_i^u cannot be continuous. If it were continuous, then, by Theorem 1 of Chen et al. (2010), the lower strategic convergence property would hold. From Theorem 1 of Chen et al. (2010), one actually obtains the following corollary of Proposition 6.[15]

Corollary 8 *If the Θ-based abstract-type-space model $\{T_i, b_i\}_{i=1}^I$ satisfies Condition 5 for agent i, it exhibits the lower strategic convergence property for agent i, i.e., the minimal $\varepsilon \geq 0$ such that in all games with the given exogenous data all actions are interim correlated ε-rationalizable for agent i with type t_i depends c ontinuously on t_i.*

[15] On can also use arguments from Engl (1995) to prove this corollary directly, For abstract type spaces, Engl shows that, if beliefs have the topology of set-wise convergence, then the Nash equilibrium correspondence has the desired lower hemi-continuity property. His arguments are easily extended to the correspondence of interim ε-rationalizable actions. Since the topology of setwise convergence is coarser than the topology induced by the total-variation metric, the lower hemi-continuity property also holds if beliefs are endowed with the latter topology.

For models with countable type spaces, an alternative way to obtain the lower strategic convergence property would be to endow type spaces with the discrete topology and to endow the spaces of beliefs with the topology of weak convergence. For a countable type space, this approach is equivalent to Condition 5. More generally though, it seems easier to assume that $T_1, ...T_I$ are compact metric spaces and to endow belief spaces with the topology that is induced by total-variation metric.

An important question is whether the topology induced by the total-variation metric is actually the coarsest topology on $\mathcal{M}(\Theta \times T_{-i})$ for which the canonical mapping from T_i to U_i^u is continuous. Or can one obtain the continuity in Proposition 6 for a coarser topology (that would of course have to be finer than the topology of weak convergence)? An answer to this question would complete the program of recovering the qualitative results of Mertens and Zamir (1985) in a setting where beliefs of an arbitrarily high order can make a significant difference to strategic behaviour.

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