



**Serial Dictatorship: the
Unique Optimal Allocation
Rule when Information is
Endogenous**

Sophie Bade





Serial Dictatorship: the Unique Optimal Allocation Rule when Information is Endogenous

Sophie Bade

May 2012

Serial Dictatorship: the Unique Optimal Allocation Rule when Information is Endogenous.

SOPHIE BADE*[†]

May 4, 2012

Abstract

The study of matching problems typically assumes that agents precisely know their preferences over the goods to be assigned. Within applied contexts, this assumption stands out as particularly counterfactual. Parents typically do invest a large amount of time and resources to find out which school would be best for their children, doctors run costly tests to establish which kidney might be best for a given patient. In this paper I introduce the assumption of endogenous information acquisition into otherwise standard house allocation problems. I find that there is a unique ex ante Pareto-optimal, strategy-proof and non-bossy allocation mechanism: serial dictatorship. This stands in sharp contrast to the very large set of such mechanisms for house allocation problems without endogenous information acquisition.

KEYWORDS: Serial Dictatorship, House Allocation Problems, Endogenous Information. *JEL Classification Numbers*: C78.

*Max Planck Institute for Research on Collective Goods, Kurt-Schumacher-Str. 10, D-53113 Bonn, Germany.

[†]Thanks to Martin Hellwig, Michael Mandler, and Philipp Weinschenk, audiences at the European Network on Matching conference in Brussels, the Jahrestagung of the Verein fuer Socialpolitik in Frankfurt, the Paris-Cologne meeting, and the University of Bonn.

1 Introduction

Many allocation problems of indivisible goods have to be solved without explicit markets. For some such goods, be it school slots or kidneys, the use of markets to determine allocations is perceived as immoral or repugnant. In many cases markets are explicitly forbidden. The subfield of mechanism design, which addresses the question on how to best allocate such objects to recipients, has prospered and many mechanisms that are optimal according to a host of different criteria have been found. These mechanisms have usually been designed for the case of agents precisely knowing their preferences over the goods to be assigned. However, this assumption seems particularly counterfactual in many of the contexts to which these mechanisms have been applied. Parents typically invest a significant amount of time on school choice; doctors need to run costly tests on kidneys to figure out which would be best for a given patient.

This paper sets out to study the allocative properties of mechanisms in conjunction with their impact on the agents' incentives to acquire information. The agents' decision to become informed about the objects to be assigned should be modeled together with their optimal choices for some given amount of information. To this end I modify the standard model of house allocation problems in which a set of agents needs to be matched to a set of equally many objects, henceforth called houses, allowing for costly information acquisition on these houses. The goal is to characterize the set of strategy-proof, non-bossy and Pareto-optimal mechanisms in this environment.

Over the years, various classes of such mechanisms have been identified for the case of the agents' preferences being given. Pycia and Unver [8] characterize the – very large – set of all such mechanisms. Lots of room remains to impose additional requirements to select among these mechanisms. The case of housing problems with endogenous information acquisition differs sharply. In that case there is a unique strategy-proof, non-bossy and ex-ante-Pareto-

optimal mechanism: serial dictatorship.¹ The following example illustrates the outstanding role of serial dictatorship.

Example 1 Consider a simple allocation mechanism for two agents to obtain some intuition for the outstanding role played by serial dictatorship. Let agents 1 and 2 start out owning two houses, k and g , respectively; this initial allocation only changes if both agents agree to exchange the houses. This mechanism is Gale’s top trading cycles mechanism for two agents and two houses; a formal definition can be found in Section 2.2. In an environment without endogenous learning, this mechanism is strategy-proof, non-bossy and Pareto-optimal. Each agent’s valuation of house k is an independent draw from the distribution, according to which it is equally likely that the agents’ value of house k is 8 or 0. Both agents value house g at 2. The agents only differ in their cost of learning: agent 1 has to pay .8 to learn his value of house k , whereas learning is free for agent 2. Assume furthermore that both agents need to announce simultaneously whether they would like to exchange houses.

Now let’s consider agent 1’s decision problem. If he does not learn the value of house k , he prefers to keep it (expected value of 4 vs. 2, the known value of g). If he learns the value, he would like to exchange with agent 2 if and only if he values house k at 0. Agent 2, in turn, is willing to exchange with a probability of $\frac{1}{2}$.² If agent 1 learns his value of house k , he obtains an expected utility of $\frac{1}{2} \times 8 + \frac{1}{2}(\frac{1}{2} \times 2 + \frac{1}{2} \times 0) - .8 = 3.7$, with the last term reflecting agent 1’s cost of learning. However, not learning is associated with an expected utility of 4 for player 1. So agent 1 keeps house k in equilibrium, which in turn implies that agent 2 is stuck with house g , yielding an ex ante utility profile of (4, 2). No consider serial dictatorship with agent 1 as the first dictator and observe that this alternative mechanism ex-ante Pareto-dominates the given mechanism. In the serial dictatorship it is worthwhile

¹Note that this statement refers to ex-ante Pareto-optimality instead of Pareto optimality. The reason is that agents do not a priori know their preferences, so Pareto optimality, which uses these preferences, cannot be the correct measuring stick. Instead, I use ex-ante Pareto optimality which also takes the learning decisions of agents into account.

²Since learning is costless agent 2 will be willing to exchange with agent 1 if and only if he values house k at 8, which happens with probability $\frac{1}{2}$.

for agent 1 to learn the value of house k and to choose it if and only if he finds it of high value (expected utility: $\frac{1}{2} \times 8 + \frac{1}{2} \times 2 - .8 = 4.2$). So agent 2 faces a probability of $\frac{1}{2}$ of ending up with either house. The profile of ex ante utilities is $(4.2, 3)$.

This example shows that some of the bedrock of the theory on house allocation problems starts to crumble once one allows for endogenous information acquisition. Both mechanisms described in the example, the top trading cycles mechanism and serial dictatorship, are Pareto-optimal, strategy-proof and non-bossy in an environment without endogenous information acquisition. Moreover, the two mechanisms are in a sense “equivalent” as shown by Abdulkadiroglu and Sonmez [1]. With endogenous information acquisition, something very different happens: Serial dictatorships not only differ from top trading cycles as allocation rules, Example 1 provides a case of a house allocation problem in which a top trading cycles mechanism is strictly ex ante Pareto-dominated by a serial dictatorship. The main result of the paper significantly generalizes Example 1. In particular, I show that for any non-bossy and strategy-proof mechanism that is not itself a serial dictatorship one can find a housing problem and a (path-dependent) serial dictatorship, such that the serial dictatorship strictly ex ante Pareto-dominates the named mechanism in the given housing problem. Conversely, serial dictatorships are never dominated in this way.

The essential difference between the two mechanisms in Example 1 is that the strong incentives for learning in serial dictatorship are dampened in top trading cycles. While agent 1’s knowledge of the value of house k is always valuable under serial dictatorship, the same knowledge is irrelevant in half of all cases in the alternative mechanism. Serial dictatorship stands out as the only mechanism which always combines optimal learning incentives with optimal allocation incentives. It is well known that serial dictatorship sets the “right” incentives for allocations: serial dictatorship is strategy-proof and Pareto-optimal. What stands out here is that it is the only mechanism that at the same time optimally incentivizes information acquisition. Any agent who acquires information knows the exact set he is choosing from at the time he does acquire information. This ensures that information is never wastefully acquired. The paper gives two variants of the uniqueness statement on serial

dictatorship pertaining to the case of sequential and simultaneous learning, Theorems 1 and 2.

While I do allow for a large set of possible information structures to obtain this result, I disallow certain conditions which cause some well-documented problems for the design of matching mechanisms. I rule out indifferences. A deviation from strict preferences is also problematic when one does not consider endogenous information acquisition. Ehlers [4] argues in this respect that “one cannot go much beyond strict preferences if one insists on efficiency and group strategy-proofness.” In addition, I assume that the agents’ preferences are independent draws. If one was to allow for correlated preferences any mechanism with sequential announcements would turn into a game of signalling. In sum, the uniqueness result does not exploit arguments that pertain to these two well-known trouble makers for the theory of house allocation problems.

The compromise between optimal information acquisition and optimal allocations is one of the main themes of the growing literature on mechanism design with endogenous information acquisition. Mechanisms are often characterized in terms of the optimal trade-off between the two criteria. Gerardi and Yariv [5] as well as Bergemann and Valimaki [2] respectively illustrate this trade-off in voting and auctions environments. This trade-off is relevant in the present paper: under a simple serial dictatorship allocative and informative efficiency coexist, it is the one mechanism under which the designer can avoid the trade off. The optimality of sequential learning is another theme of the literature on mechanisms with endogenous learning that is echoed in the present paper. Gershkov and Szentes [6] as well as Smorodinsky and Tennenholtz [11] present voting models in which the voters’ optimal acquisition of information is sequential. Similarly, for auctions, Compte and Jehiel [3] find that ascending price auctions can dominate sealed bid auctions in terms of expected welfare. In this vein the present paper shows the unique optimality of *sequential* simple serial dictatorship when allowing for *any* sequence of information elicitation.

In Section 2 I provide formal definitions of the environments and mechanisms under study. Using an example (Example 2), I argue that sequential elicitation procedures might outperform simultaneous ones in the present

context. With all the relevant terminology in hand, I state the two main results of the article Theorems 1 and 2 in Section 3. The proof of these two theorems revolves around three examples: Example 2, which is presented in Section 2, the introductory Example 1, which is revisited in Section 4, and Example 5 which is presented in the same section. After a presentation of these examples, I devote a section to the presentation of Pycia and Unver’s [8] “trading cycles” mechanisms, which constitute the appropriate starting point for the characterization of the present paper. This is followed by a section devoted to the proof of the two main results of the paper (Section 6).

2 The Model

2.1 Agents, Houses, Values

Fix two sets of agents $I = \{1, \dots, n\}$ and houses H with equally many elements ($|H| = n$) and generic elements $i, j, i', j' \in I$ and $h, d, g, k \in H$. An **environment** (for (H, I)) \mathcal{E} has three components: (Ω, π, c) . The finite state space Ω consists of profiles of values $\omega := (\omega_h^i)_{h \in H, i \in I}$, where ω_h^i is the value that agent i assigns to house h . The state $\omega \in \Omega$ is drawn from the probability distribution π . The vector $c := (c^i)_{i \in I}$ of cost functions $c^i : \mathcal{P} \rightarrow \mathbb{R}_0^+ \cup \{\infty\}$ describes the agents’ learning technologies and assigns every partition $P \in \mathcal{P}$ of the state space Ω a non-negative and possibly infinite cost. A partition P with $c^i(P) = \infty$ is called **prohibitively costly (for agent i)**. Conversely, all other partitions are called **i -affordable**, also an event E is called **i -affordable** if it is part of an i -affordable partition.

The vector $\omega^i := (\omega_h^i)_{h \in H}$ summarizes agent i ’s valuations of all houses at the given state ω . Define the algebra ζ^i on Ω as the algebra generated by all events $E^i := \{\omega \mid \omega^i = \bar{\omega}^i\}$ for some fixed $\bar{\omega}^i \in \mathbb{R}^n$. It is assumed that the agents’ preferences are independently drawn, formally, $\pi(E^i \cap E^j) = \pi(E^i)\pi(E^j)$ for all $E^i \in \zeta^i$ and $E^j \in \zeta^j$. Every state occurs with positive probability, $\pi(\omega) > 0$ for all $\omega \in \Omega$. The prior π is common knowledge among the designer and all agents. Agent i can only learn something about his own preferences ω^i ; formally, a partition P containing an event $E \notin \zeta^i$ is prohibitively costly for agent i . Next, staying ignorant is free; that is

$c^i(\{\Omega, \emptyset\}) = 0$. Define $P^i(\omega)$ as the event E that agent i knows at state ω when his information partition is P^i . Let \bar{P}^i be the partition according to which agent i knows his value for each of the houses.

The posterior value that agent i assigns to house h when he knows event E is calculated as $\bar{\omega}_h^i(E) := \frac{\sum_{\omega_h^i \in E} \pi(\omega_h^i) \omega_h^i}{\sum_{\omega_h^i \in E} \pi(\omega_h^i)}$. It is assumed that $\bar{\omega}_h^i(E) \neq \bar{\omega}_g^i(E)$ for any two houses $h \neq g$ and any i -affordable event E , so agents are never indifferent between two houses for any ex post preferences. Moreover, it is assumed that the following inequality holds for all i -affordable partitions $P \neq Q$ and all subsets of houses $S \subset H$:

$$\sum_{E \in P} \pi(E) \max_{h \in S} \bar{\omega}_h^i(E) - c(P) \neq \sum_{E \in Q} \pi(E) \max_{h \in S} \bar{\omega}_h^i(E) - c(Q).$$

This implies that any agent has a *unique* best learning choice that maximizes his ex ante utility of his – informed – choice from any set $S \subset H$. The sets of agents and houses I and H together with the environment \mathcal{E} constitute a **housing problem (with endogenous information acquisition)**, since only the size $n = |I|$ of I and H matters such a housing problem is denoted by $\langle n, \mathcal{E} \rangle$.

Mechanisms are used to obtain matchings for housing problems. A **matching** is a bijective function $\mu : I \rightarrow H$, where $\mu(i)$ is called agent i 's **assignment** under μ . A **submatching** $\sigma : I_\sigma \rightarrow H_\sigma$ is a bijective function with $I_\sigma \subsetneq I$ and $H_\sigma \subsetneq H$. Analogously $\sigma(i)$ is called agent i 's assignment under the submatching σ . The sets of all matchings and of all submatchings, respectively are denoted by \mathcal{M} and by $\bar{\mathcal{M}}$. For any particular submatching $\sigma \in \bar{\mathcal{M}}$ the sets of unmatched agents and houses are denoted as \bar{I}_σ and \bar{H}_σ respectively.

2.2 Standard Housing Problems

The set of housing problems with endogenous learning embeds the set of standard housing problems: A housing problem $\langle n, \mathcal{E} \rangle$ is a **standard housing problem** if $\mathcal{E} = (\Omega, \pi, c)$ is such that Ω is a singleton $\{\omega\}$.³ I denote

³The most general way to embed standard housing problems in the set of housing problems with endogenous information acquisition would be to demand that $\{\Omega\}$ is the unique i -affordable partition for all agents i .

a standard housing problem by $\langle n, \omega \rangle$. Let Θ be the space of all possible vectors of values of an agent (ω^i). The set of all standard housing problems (of some fixed size n) is denoted by Θ^n , the set of all profiles of values ω without any indifferences.

A **(direct) mechanism** is a mapping $\varphi : \Theta^n \rightarrow \mathcal{M}$ that assigns a matching for each preference profile $\omega \in \Theta^n$. Such a mechanism is considered strategy-proof if the truthful revelation of preferences is a weakly dominant strategy. A mechanism is considered non-bossy (as defined by Satterthwaite and Sonnenschein [9]) if an agent can only change the allocation of some other agent if he also changes his own allocation. This implies that any misreport of preferences that does not change the agent’s own assignment does not change anyone else’s assignment.⁴ The mechanism φ is considered Pareto-optimal if $\varphi(\omega)$ is Pareto-optimal for any ω . The set of strategy-proof, non-bossy, and Pareto-optimal mechanisms has been characterized by Pycia and Unver [8]. Since this characterization plays a crucial role for the development of the arguments in the present article, I present it in Section 5. I next define three canonical examples of strategy-proof, non-bossy, and Pareto-optimal matching mechanisms.

Serial dictatorships and top trading cycles play an outstanding role in the theory of matching. According to a **simple serial dictatorship**, one agent, the first dictator, gets to choose a house out of the grand set of houses H . Next, another agent, the second dictator, gets to choose a house out of the remainder, and so forth, until all houses are matched. I denote a simple serial dictatorship as a direct mechanism by $\delta : \Theta^n \rightarrow \mathcal{M}$. The bijection $[\delta] : I \rightarrow \{1, \dots, n\}$ denotes the ordering of the dictators in the sense that agent i is the $[\delta](i)$ -th dictator under the serial dictatorship δ .

The reason for the prefix “simple” arises since **path-dependent serial dictatorships** also play a role in the present paper. According to this type of serial dictatorship, the identity of the dictator in some round depends

⁴Papai [7] shows that a direct mechanism (for standard house allocation problems) is strategy-proof and non-bossy if and only if it is group strategy-proof, in the sense that no group of agents can misreport their preferences to make at least one member of the group better off without hurting any member in the group. Takamiya [12] shows that group strategy-proofness is equivalent to Maskin monotonicity in the context of standard housing problems.

on the houses chosen by the preceding dictators. A path-dependent serial dictatorship is denoted by a function γ that maps any vector of already chosen houses to the agent who becomes the dictator after this sequence of choices. A simple serial dictatorship is path-dependent serial dictatorship with the special property that the identity of the dictator only depends on the length of the vector of houses already chosen.

Gale’s top trading cycles mechanism⁵ is the other canonical example of a matching mechanism. This mechanism starts out with a matching ρ called the initial endowment. All agents are asked to point to the agent who has been endowed with their most preferred house. At least one cycle forms. All agents in such cycles are assigned the houses that they point to. The procedure is repeated until all houses are assigned. I denote Gale’s top trading cycles mechanism, starting with the endowment ρ by τ^ρ . The mechanism discussed in Example 1 is a top trading cycles mechanism.

2.3 Dynamic Direct Revelation Mechanisms

In this section I define the grand set of mechanisms considered in the present article together with a list of canonical examples. Let me first use the example of serial dictatorship to argue that the sequence of preference announcements can be essential in direct revelation mechanisms with endogenous information acquisition.

Example 2 Two different dynamic versions of serial dictatorship stand out: The designer might either simultaneously elicit the preferences of all agents; alternatively the designer might elicit the preferences of some agents only once he becomes a dictator – and thereby leave him the choice to tailor his information acquisition to his actual choice set. To see that this difference matters, consider a housing problem $\langle 3, \mathcal{E}^b \rangle$ with $H = \{d, g, k\}$, $\mathcal{E}^b = (\Omega, \pi, c)$, such that the three agents are a priori identical with the value of house d being either 8 or 0, each with probability $\frac{1}{2}$, and the values of houses g and k known to be 5 and 2, respectively. Assume that it costs each agent $c = .1$ to learn his type. If the designer simultaneously elicits preferences, it is

⁵This mechanism was first defined by Shapley and Scarf [10], who attribute it to David Gale.

worthwhile for the first and second dictators to learn their type. However, if the designer elicits preferences sequentially, then the second dictator will only learn his type if the first did not choose house d . The sequential mechanism ex ante Pareto-dominates the mechanism of simultaneous elicitation, as the second dictator will avoid acquiring information (and thereby save the costs $c = .1$) in the case in which it is of no consequence to his decision.

Since the timing of announcements greatly matters in mechanism design problems with endogenous information acquisition, I consider the set of all **dynamic (direct) mechanisms**. The designer can fix an order of the agents' announcements. This order is described by a rooted tree t that describes the agents' communication to the designer, called a **c-tree**. The initial node of a c-tree is labeled with the first agent to declare a preference. The next agent to declare a preference is allowed to depend on the declaration of the prior agent(s); each branch terminates with the declaration of all agents's types. Formally the space of possible declarations Θ by the first agent (w.l.o.g.) 1 is divided into $n - 1$ subsets $\Theta_2, \dots, \Theta_n$. If agent 1's announcement belongs to the subset Θ_j , then agent j is the next one to announce. Inductively the space of the announcements of the preceding m is partitioned into subspaces $(\Theta \times \dots \times \Theta)_2, \dots, (\Theta \times \dots \times \Theta)_n$ such that agent j gets to announce his preference after any sequence of announcements $(\Theta \times \dots \times \Theta)_j$ and such that no agent gets to announce their ranking twice. The designer can freely choose the sequencing of announcements as well as the information sets on the c-tree t . The dynamic mechanism induced by the c-tree t , and the direct mechanism φ , is denoted as $\langle \varphi, t \rangle$.

Together with an environment \mathcal{E} and an assumption on learning the dynamic direct revelation mechanism, $\langle \varphi, t \rangle$, induces an extensive form game $\langle \varphi, t \rangle(\mathcal{E})$. This game starts with a chance node in which nature determines the state of the world ω . Agents get to declare their preferences according to the c-tree t . Before each node in which an agent gets to declare his preference, he can choose an information partition on the state space Ω . The information sets in the extensive form game reflect the privacy of learning as well as the information sets established through the c-tree t . Utilities are calculated based on the expected utility of the house the agent is assigned minus the information-acquisition cost the agent incurred.

Of course, in many contexts, sequential learning might be impractical. This is the case when there are very many agents or when learning takes up much time. I therefore also study the subclass of dynamic mechanisms in which the designer simultaneously elicits all preferences. Formally a mechanism $\langle \varphi, t^s \rangle$ is defined as a **simultaneous (direct) mechanism** where t^s is the c-tree according to which no agent knows anything about the other agents' announcements when he announces his own preferences.

Next, define a **sequential simple serial dictatorship** or **3S dictatorship** as the dynamic direct revelation mechanism $\langle \delta, t^\delta \rangle$, where t^δ is the c-tree, according to which agents announce their preference according to their ordering as dictators in δ (the n -th dictator knows the preference-announcement of all $n - 1$ preceding dictators when it is his turn to announce his preferences). Analogously a dynamic direct revelation mechanism is a **sequential path-dependent serial dictatorship** (γ, t^γ) if t^γ is such that agents announce their preferences in the same sequence as the sequence in which they do become dictators.

2.4 Equilibria and Outcomes

A (mixed) strategy profile in $\langle \varphi, t \rangle(\mathcal{E})$ is considered an **equilibrium** if it is a perfect Bayesian equilibrium and if agents truthfully announce their types in the sense that any agent i with ex post preferences $\bar{\omega}^i$ announces these preferences to the designer. In the standard case it is straightforward to see that there exists at most one equilibrium. In that case each agent knows his ranking ω^i , the question is just whether telling it is an equilibrium. In the present case there might be multiple equilibria, since the agents' ex post rankings depend on the agents' informational choices. I next give an example of an environment \mathcal{E} together with a mechanism $\langle \varphi, t \rangle$ such that $\langle \varphi, t \rangle(\mathcal{E})$ has multiple equilibria.

Example 3 Let $n = 2$ and $H = \{k, g\}$. Consider an environment $\mathcal{E} = (\Omega, \pi, c)$ as follows: Ω has 4 equiprobable states. Agent 1's valuation of house k might either be 8 or 0; he is sure to value house g at 3. Conversely, agent 2's valuation of house g might either be 8 or 0; he is sure to value house k at 3. It costs each agent .1 to find out his preference. Consider Gale's top trading

cycles mechanism τ^ρ with the initial endowment ρ , according to which agent 1 starts out owning house k . The game $\langle \tau^\rho, t^s \rangle(\mathcal{E})$ (in which both agents need to announce their rankings simultaneously) has two equilibria. According to the first, neither agent learns anything and always points to the house he was endowed with. According to the other, both agents learn their true values and point to the house they find to be of higher value. Note that in either one of these equilibria the agents tell the truth.

Any strategy profile in $\langle \varphi, t \rangle(\mathcal{E})$ is associated with an outcome function $f : \Omega \rightarrow (\mathcal{M}, \mathcal{P})$ which maps any state ω to a profile of information partition $(P^i[\omega])_{i \in I}$ and a matching $\mu[\omega] \in \mathcal{M}^6$. A mechanism $\langle \varphi, t \rangle$ is said to **implement** an outcome function f at the environment \mathcal{E} if $\langle \varphi, t \rangle(\mathcal{E})$ has an equilibrium strategy profile that is associated with the outcome function f . The set of all outcome functions implemented by $\langle \varphi, t \rangle$ at \mathcal{E} is denoted by $o(\langle \varphi, t \rangle(\mathcal{E}))$. The ex-ante utility U^i of agent i associated with a given outcome function $f : \Omega \rightarrow (\mathcal{M}, \mathcal{P})$ is defined as

$$U^i(f) = \sum_{\omega \in \Omega} \pi(\omega) \left(\bar{w}_{\mu[\omega](i)}^i(P^i\omega) - c^i(P^i(\omega)) \right).$$

One outcome function f is said to ex ante Pareto-dominate another outcome function f' if $U^i(f) \geq U^i(f')$ holds for all $i \in I$ and if $U^j(f) > U^j(f')$ holds for some $j \in I$. If *all* outcome functions implemented by $\langle \varphi, t \rangle(\mathcal{E})$ ex ante Pareto-dominate *all* outcome functions implemented by a different dynamic direct revelation mechanism $\langle \varphi', t' \rangle$ at \mathcal{E} , then I write $\langle \varphi, t \rangle(\mathcal{E}) \succ^* \langle \varphi', t' \rangle(\mathcal{E})$. I then say that mechanism $\langle \varphi, t \rangle$ ex ante Pareto-dominates mechanism $\langle \varphi', t' \rangle$ at environment \mathcal{E} .

3 The Uniqueness of Serial Dictatorship

It is the goal of this article to characterize all strategy-proof and non-bossy mechanisms φ with c-trees t such that the dynamic direct revelation mecha-

⁶This needs to be distinguished from $P^i(\omega)$ which is the event known by agent i at ω , when he holds the partition P^i . So at ω agent i knows the event $P^i\omega$ when he acquires the partition prescribed by the outcome function $P^i[\omega]$.

nism $\langle \varphi, t \rangle$ is ex ante Pareto-optimal for any housing problem with endogenous information acquisition. The next two theorems show that simple serial dictatorship is the only such mechanism - whether one allows for all dynamic direct revelation mechanisms or only for the simultaneous ones.

Theorem 1 1) Fix any environment \mathcal{E} . No outcome function implemented by some dynamic direct revelation mechanism, $f \in o(\langle \varphi, t \rangle(\mathcal{E}))$, ex ante Pareto-dominates the outcome function implemented by 3S dictatorship $o(\langle \delta, t^\delta \rangle(\mathcal{E}))$.

2) For any strategy-proof and non-bossy dynamic direct revelation mechanism that is not a 3S dictatorship, there exists an environment and a path-dependent serial dictatorship such that any outcome function implemented by the given dynamic direct revelation mechanism is Pareto-dominated by the outcome function implemented by the sequential path-dependent serial dictatorship. Formally, for all $\langle \varphi, t \rangle$ with φ strategy-proof and non-bossy there exists a path-dependent serial dictatorship γ and an environment \mathcal{E} , such that $\langle \gamma, t^\gamma \rangle(\mathcal{E}) \succ^* \langle \varphi, t \rangle(\mathcal{E})$.

An important asymmetry should be noted between the first and the second part of the theorem: the first part says that simple serial dictatorship goes undominated. However, not all mechanisms can be dominated by serial dictatorships: the second part of the theorem refers to path-dependent serial dictatorships. If one restricts attention to simultaneous direct revelation mechanisms (φ, t^s) , a nearly identical result holds:

Theorem 2 1) Fix any environment \mathcal{E} . No outcome function implemented by some simultaneous direct revelation mechanism, $f \in (\langle \varphi, t^s \rangle(\mathcal{E}))$, ex ante Pareto-dominates all outcome functions implemented by simultaneous simple serial dictatorship $o(\langle \delta, t^s \rangle(\mathcal{E}))$.

2) For any strategy-proof and non-bossy simultaneous direct revelation mechanism that is not a simultaneous simple serial dictatorship, there exists an environment and a path-dependent serial dictatorship such that the outcome function implemented by the simultaneous path-dependent serial dictatorship Pareto-dominates any outcome function implemented by the given simultaneous direct revelation mechanism. Formally, for all $\langle \varphi, t^s \rangle$ with φ

strategy-proof and non-bossy there exists a path-dependent serial dictatorship γ and an environment \mathcal{E} , such that $\langle \gamma, t^s \rangle(\mathcal{E}) \succ^ \langle \varphi, t^s \rangle(\mathcal{E})$.*

Note that the two theorems are nearly identical up to a switch of “dynamic” and “simultaneous” in all the relevant places (including a replacement of the c-trees t and t^δ for the dynamic mechanisms with the simultaneous c-tree t^s). However, there is another – minor – difference which concerns the first part of each theorem. In Theorem 1 it is claimed that no dynamic direct revelation mechanism Pareto-dominates *any* outcome of 3S dictatorship. Differently, Theorem 2 makes the corresponding statement with respect to *all* outcomes of simultaneous simple serial dictatorship. The reason for this difference is that 3S dictatorship always has a unique outcome, whereas simultaneous simple serial dictatorship might have multiple equilibria. Before going into any further detail, let me discuss two examples, that on the one hand, illustrate the two theorems and, on the other hand, constitute the backbone of their proof.

4 Two Examples

This section provides two examples of very small housing problems together with dynamic direct revelation mechanisms $\langle \varphi, t \rangle$ that are not 3S dictatorships. In either case I present an environment \mathcal{E} , such that the mechanism $\langle \varphi, t \rangle$ is ex ante Pareto-dominated by some sequential path-dependent serial dictatorship γ at \mathcal{E} : formally $\langle \gamma, t^\gamma \rangle(\mathcal{E}) \succ^* \langle \varphi, t \rangle(\mathcal{E})$. These examples show that Theorems 1 and 2 hold for the special mechanisms provided in the two examples. The proof of Theorems 1 and 2 relies on generalizing the insights gained in these two examples and Example 2 to the large set of dynamic direct choice mechanisms.

Example 4 Here I revisit Example 1 to argue that part 2 of Theorems 1 and 2 holds for $n = 2$. To see this, observe that with only two agents there is only one strategy-proof and Pareto-optimal mechanism that is not a serial dictatorship: fixing an initial endowment and letting agents trade freely, as defined in Example 1. Observe that at least one agent in this mechanism needs to declare his preferences to the designer before knowing the preference

of the other. Assume that this is agent 1. To see that there is an environment, such that serial dictatorship ex ante Pareto-dominates the given mechanism, consider the environment defined in Example 1.

With the formalism developed so far, this environment \mathcal{E}^a can be described succinctly as follows: Take $n = 2$ and $H = \{k, g\}$, let $\pi(\omega_k^i = 8) = \pi(\omega_k^i = 0) = \frac{1}{2}$ and $\pi(\omega_g^i = 2) = 1$ for $i = 1, 2$, $c^1(\bar{P}^1) = .8$ and $c^2(\bar{P}^2) = 0$.⁷ As argued in the introduction, agent 1's costs of learning outweigh his benefit of learning if he starts out as the owner of house k in the trading mechanism: in equilibrium there is no exchange in this mechanism, yielding the ex ante utility profile $(4, 2)$. Conversely, if agent 1 is the first dictator, learning is worthwhile for him; in this case, agent 2 has a chance to obtain the ex-ante more valuable house g , the ex-ante utility profile of this serial dictatorship is $(4.2, 3)$. Observe that I made only one reference to the sequence of announcements: In the trading mechanism, (at least) one agent needs to announce his preferences before knowing the preferences of the other. As this holds for simultaneous and sequential versions of the mechanism and as the outcome of serial dictatorship depends on only one announcement when there are just two agents, the above shows part 2 of Theorems 1 and 2 for $n = 2$.

Observe that with just 2 agents path-dependent and simple serial dictatorship coincide. In the next example I show that with 3 agents path-dependent serial dictatorships can be ex-ante dominated by other path-dependent serial dictatorships at some environments.

Example 5 Take $n = 3$ and $H = \{g, k, d\}$. Consider the sequential path-dependent serial dictatorship $\langle \gamma, t^\gamma \rangle$ with agent 1 as the first dictator. If he chooses g , then agent 2 gets to choose from $\{k, d\}$; otherwise, agent 3 becomes the next dictator. Define the environment \mathcal{E}^c , such that agent 1's utility vector for the three houses ω^1 is either $(2, 1, 0)$ or $(0, 2, 1)$ - each with probability $\frac{1}{2}$. Agent 1 faces a cost of .1 to learn his type. The utility vectors of agent 2 and 3 are known to be $\omega^2 = (10, 2, 0)$ and $\omega^3 = (2, 10, 0)$, respectively. So $\langle \gamma, t^\gamma \rangle$ has a unique equilibrium, the unique vector of ex ante

⁷Nothing of substance would change if one was to assume that it costs the other $0 < c^2 < .5$ to learn his value of house k . This case is tedious since one has to solve for a mixed strategy equilibrium.

utilities $(1.9, 1, 1)$. Now consider the alternative sequential path-dependent serial dictatorship (γ', t') which also starts with agent 1 as the first dictator, but then continues with 3 as the next dictator if agent 1 chose g , and agent 2 otherwise. The vector of ex ante utilities implemented by $(\gamma', t')(\mathcal{E}^c)$ is $(1.9, 5, 5)$.

Note that timing of announcements does not matter for the given environment; formally $o(\langle \gamma, t \rangle(\mathcal{E}^c))$ does not depend on t . The reason is that there is only agent who has any information to acquire. Also note that any path-dependent serial dictatorship that is not a simple serial dictatorship – up to renaming – equivalent to the mechanism γ defined in the above example. In sum, the example shows that part 2 of Theorems 1 and 2 is true when one only considers the case of path-dependent serial dictatorships and $n = 3$.

5 The Trading Cycles Mechanism

The proof of Theorem 1 starts by restricting the set of direct revelation mechanisms φ to be considered. For $\langle \varphi, t \rangle$ to be strategy-proof, non-bossy, and ex ante Pareto-optimal, the direct revelation mechanism φ must itself be non-bossy, strategy-proof and Pareto-optimal in the subset of standard housing problems. The set of such mechanisms φ has been characterized by Pycia and Unver [8] as the set of trading cycles mechanisms. In trading cycles mechanisms, just like in Gale’s top trading cycles mechanism, there is an initial allocation of all houses to the agents, and assignments are then determined through trade in cycles. Trading cycles mechanisms generalize Gale’s top trading cycles mechanism in two major ways: First of all, agents can own more than one house before they leave with their assignment. Once an owner of multiple houses leaves the mechanism his as of yet unmatched houses are passed on to the remaining agents via a fixed inheritance rule.⁸ Secondly, there are two types of control rights in trading cycles mechanisms, in addition to ownership, as in Gale’s top trading cycles mechanism there is

⁸This first difference between Gale’s top trading cycles and the trading cycles mechanism already appears in Papai [7].

a new form of control called brokerage. A broker can exchange the house he controls for a different house; however, he may not himself appropriate the house.

Formally, any **trading cycles** mechanism following Pycia and Unver [8] is fully defined through a **control rights structure** (c, b) . Such a structure assigns control rights for any submatching σ : $(c, b) = \{(c_\sigma, b_\sigma) : \overline{H}_\sigma \rightarrow \overline{I}_\sigma \times (ow, br)\}_{\sigma \in \overline{\mathcal{M}}}$ with $c_\sigma(h)$ the agent controlling house h and $b_\sigma(i)$ the type of control at submatching σ (br for brokerage, and ow for ownership). A control rights structure (c, b) is considered **consistent** if it satisfies the following requirements R1-R6, which I take word by word from Pycia and Unver [8]:

Within round requirements. Consider any $\sigma \in \overline{\mathcal{M}}$:

- (R1) There is at most one brokered house at σ .
- (R2) If i is the only unmatched agent at σ , then i owns all unmatched houses at σ .
- (R3) If agent i brokers a house at σ , then i does not own any houses at σ .

Across-round requirements. Consider any submatchings σ, σ' , such that $|\sigma'| = |\sigma| + 1$ and $\sigma \subset \sigma' \in \overline{\mathcal{M}}$, and any agent $i \in \overline{I}_{\sigma'}$ and any house $h \in \overline{H}_{\sigma'}$:

- (R4) If i owns h at σ then i owns h at σ' .
- (R5) Assume that at least two agents from \overline{I}_σ own houses at σ' . If i brokers house h at σ then i brokers h at σ' .
- (R6) Assume that at σ agent i controls h and agent $i' \in \overline{I}_\sigma$ controls $h' \in \overline{H}_\sigma$. Then, i' owns h at $\sigma \cup \{(i, h')\}$, and if, in addition, i brokers h at σ but not at σ' and $i' \in \overline{I}_{\sigma'}$, then i' owns h at σ' .

The following algorithm, consisting of a finite sequence of rounds $r = 1, 2, \dots$ establishes the outcome of the trading cycles mechanism $\psi^{c,b}$: The submatching of agents and houses matched before round r is denoted by σ^{r-1} , with $\sigma^0 = \emptyset$. In round r each house $h \in \overline{H}_{\sigma^{r-1}}$ points to the agent who controls it at σ^{r-1} . If there exists a broker at σ^{r-1} , then he points to his most preferred house among the ones owned at σ^{r-1} . All other agents point to their most preferred house in $\overline{H}_{\sigma^{r-1}}$. There exists at least one cycle of agents and houses pointing to each other. Each agent in each such trading cycle is matched with the house he is pointing to. The union of σ^{r-1} and

the newly matched agent-house pairs defines σ^r . If σ^r is a matching, the mechanism terminates. Pycia and Unver [8] show that a direct revelation mechanism φ is strategy-proof, non-bossy and Pareto-optimal if and only if there exists a consistent control rights structure (c, b) such that $\varphi = \psi^{c,b}$. A discussion of the trading cycles mechanism goes beyond of the scope of the present paper and can be found in their paper. Here I just wish to note that path-dependent serial dictatorships (for each σ there exists an i_σ such that $c_\sigma(h) = i_\sigma$), simple serial dictatorships ($i_\sigma = i_{\sigma'}$ if $|\sigma| = |\sigma'|$ has to hold in addition to the preceding requirement), and Gale's top trading cycles (there exists a matching μ such that $c_\emptyset(h) = \mu^{-1}(h), b_\sigma(h) = ow$ for all $h \in H$) are special cases of trading cycles mechanisms.

6 Proof of Theorems 1 and 2

Observe first of all that the claim has been shown for $n = 2$ by Example 4. So, assume throughout that I, H with $n \geq 3$ are fixed. The proof of part 1) of Theorem 1 uses the following result on the unique equilibrium outcome of any sequential path-dependent serial dictatorship, which is essentially derived from the no-indifference assumptions on the environments \mathcal{E} under study.

Lemma 1 *Any sequential path-dependent serial dictatorship $\langle \gamma, t^\gamma \rangle$ implements a unique outcome function at any environment \mathcal{E} ; $o(\langle \gamma, t^\gamma \rangle(\mathcal{E}))$ is a singleton.*⁹

To see that 3S dictatorship can never be ex ante Pareto-dominated fix some 3S dictatorship $\langle \delta, t^\delta \rangle$ together with an environment \mathcal{E} . Suppose that some other mechanism $\langle \varphi, t \rangle$ has an ex ante Pareto dominating outcome at \mathcal{E} . It cannot be that the first dictator $[\delta]^{-1}(1)$ is any better off under $\langle \varphi, t \rangle(\mathcal{E})$. The first dictator has a unique optimal and truthful strategy and cannot be made any better off by a different mechanism. Now observe that the uniqueness of best strategies implies again that second dictator $[\delta]^{-1}(2)$ cannot be made any better off conditional on the distribution of choice sets associated with the first dictator's unique equilibrium strategy. This argument can be

⁹The proof of all Lemmas appears in the Appendix.

applied inductively to all following dictators. This proves part 1) of Theorem 1.

To prove part 2) of Theorem 1, I subdivide the set of dynamic direct mechanisms $\langle \varphi, t \rangle$ that are not 3S dictatorships into three categories: I) φ is not a path-dependent serial dictatorship, II) φ is a path-dependent serial dictatorship without being a simple serial dictatorship, III) φ is a simple serial dictatorship δ , but t is not equal to t^δ . Based on the work of Pycia and Unver [8], I show that any strategy-proof, non-bossy and Pareto-optimal φ that is not a path-dependent serial dictatorship is a trading cycles mechanism with at least two owners in some round (Lemma 2). With this observation in mind, one can see that for each of the three subclasses I already provided a generic example of a mechanism that belongs to this class together with an environment \mathcal{E} , such that the exemplary mechanism is ex ante Pareto-dominated by some path-dependent serial dictatorship at \mathcal{E} . In Examples 1 and 4 I discussed the example of a trading cycles mechanism with two owners, and defined an environment \mathcal{E}^a at which this mechanism is ex ante Pareto-dominated by an (appropriately defined) 3S dictatorship. Example 5 covers the case of a path-dependent serial dictatorship that is not a simple serial dictatorship in a housing problem with just 3 agents. In Example 2 I discussed a dynamic simple serial dictatorship (δ, t) that is not sequential and showed that this mechanism is ex ante Pareto-dominated by a 3S- dictatorship at the environment \mathcal{E}^b . In Lemmas 3, 4, and 5 I use the arguments made with respect to these three small scale examples in the context of housing problems with a large number of agents and houses. The technique of proof in all three cases is to embed the environment constructed in the examples into an environment for a housing problem of any size.

Lemma 2 *Consider a trading cycles mechanism $\psi^{c,b}$, if $\psi^{c,b} \neq \gamma$ for γ any path-dependent serial dictatorship. Then there exists a submatching σ and two houses d, g , such that $(c_\sigma, b_\sigma)(d) = (i, ow)$ and $(c_\sigma, b_\sigma)(g) = (j, ow)$ with $i \neq j$.*

Together with the characterization by Pycia and Unver, this implies that any strategy-proof, non-bossy and Pareto-optimal direct revelation mecha-

nism φ is either a path-dependent serial dictatorship or is a trading cycles mechanism with at least two owners at some round.

Lemma 3 *Fix any control rights structure (c, b) and assume that $\psi^{c,b}$ has a round with at least two owners. Fix any c-tree t . There exists an environment \mathcal{E} and serial dictatorship δ such that any outcome of $\langle \varphi, t \rangle(\mathcal{E})$ is ex ante Pareto-dominated by the outcome of $\langle \delta, t^\delta \rangle(\mathcal{E})$.*

Lemma 4 *Fix a path-dependent serial dictatorship γ , that is not a simple serial dictatorship, together with a c-tree t . There exists an environment \mathcal{E} and another path-dependent serial dictatorship γ' such that the unique outcome of $\langle \gamma', t' \rangle(\mathcal{E})$ Pareto-dominates the all outcomes of $\langle \gamma, t \rangle(\mathcal{E})$.*

Lemma 5 *Fix a serial dictatorship δ together with a c-tree $t \neq t^\delta$. There exists an environment \mathcal{E} such that the outcome of the sequential serial dictatorship $(\delta, t)(\mathcal{E})$ is ex ante Pareto-dominated by the outcome of the 3S dictatorship $\langle \delta, t^\delta \rangle(\mathcal{E})$.*

Observe that Lemma 3 was proven by Example 1 for the case of $n = 2$, and that Lemmas 4 and 5 were proven by Examples 5 and 2, respectively, for the case of $n = 3$. The proof of these Lemmas for $n > 3$ consists in embedding the environments defined in these examples into environments with n houses and agents. Also observe that in sum Lemmas 2 through 5 constitute the proof of part 2 of Theorem 1. These Lemmas show that for any conceivable deviation from 3S dictatorship there exists an environment such that some path-dependent serial dictatorship dominates that mechanism at this environment.

The proof of Theorem 2 follows from the observation that the sequentiality of announcements neither matters for the arguments brought forward with respect to Examples 5 and 4, nor for their embedding in larger environments. The environments \mathcal{E} chosen to prove Lemmas 3 and 4 are defined such that in all relevant cases the outcome of the relevant direct mechanism φ together a simultaneous c-tree t^s coincides with the outcome of $\langle \varphi, t \rangle$ for any other c-tree. The proof of Theorem 2 can be found in the Appendix.

7 Conclusion

If one allows for endogenous information acquisition in housing problems, simple serial dictatorships stand out from the large set of strategy-proof, non-bossy and Pareto-optimal mechanisms. Whether one looks at mechanisms that dynamically elicit preferences or only at the subset of mechanisms in which preferences are elicited simultaneously: simple serial dictatorships are the only ex-ante Pareto-optimal mechanisms.

Within the set of strategy-proof and non-bossy mechanisms, serial dictatorships are unique in the sense that they always provide optimal learning incentives. When agents need to decide on learning in a serial dictatorship they know all relevant information about their choice sets to tailor their information acquisition optimally. Example 1 shows that this is not the case for Gale's top trading cycles mechanism with just two agents: Here it might be that an agent needs to decide what to learn when he only knows the distribution over his possible choice sets. That example was constructed such that agent 1 avoids learning. In addition, agent 2 would rather award agent 1 dictator rights, to get agent 1 to learn, than to stay with the equilibrium allocation of the case that agent 1 does not learn. The main argument of the proof was that any strategy-proof, non-bossy and Pareto-optimal direct choice mechanism that is not a serial dictatorship in a sense embeds Gale's top trading cycles mechanism with just two agents.

Abdulkadiroglu and Sonmez [1] show that serial dictatorship with the order of dictators randomly drawn from a uniform distribution (random priority mechanism) is equivalent to Gale's top trading cycles mechanism with the endowment drawn from a uniform distribution. To see that this equivalence result does not hold for the case of endogenous information acquisition, reconsider the environment \mathcal{E}^a defined and discussed in Examples 1 and 4. The distribution over ex ante utility profiles for the two serial dictatorships in which agents get to learn once they know whether they are the first or second dictator is $\frac{1}{2}(3, 5) + \frac{1}{2}(4.2, 3)$, so the profile of ex-ante utilities is $(3.6, 4)$. Conversely, for the case of Gale's top trading cycles mechanism with random endowments and simultaneous announcements, we have $\frac{1}{2}(3, 5) + \frac{1}{2}(4, 2)$ as the distribution, so the profile of ex ante utilities is $(3.5, 3.5)$. So, in this ex-

ample, the random priority mechanism fares better than Gale's top trading cycles mechanism with random endowments. The question to what extent this observation can be generalized remains open.

It could also be interesting to study ex-ante Pareto optimality without the assumption of endogenous learning. This study could be couched in a version of the model considered here with $c^i(P) = 0$ for any partition P containing only elements $E \in \zeta^i$, meaning that agents face no cost of learning their own types. Observe that the cost of learning in Example 5 played no role, so the argument that any path-dependent serial dictatorship is ex-ante Pareto-dominated by another path-dependent serial dictatorship for some environment \mathcal{E} also applies to this special case.

Observe that Example 5 can be re-interpreted to see a conflict between bossiness and ex ante Pareto optimality. To see this, change the mechanism γ defined in that example to a very similar type of bossy serial dictatorship in which the identity of the second dictator does not depend on whether agent 1 chooses house g or k , but rather on whether he ranks house d at the bottom or not. Say agent 2 becomes the second dictator if and only if agent 1 ranks house d at the bottom. This is a bossy mechanism, since agent 1's assignment does not change when announcing either $(2, 1, 0)$ or $(2, 0, 1)$. However, the assignments to the following two dictators will vary with agent 1's announcement if their preferences are aligned. Now observe that for \mathcal{E}^c the environment given in Example 5 this bossy mechanism is essentially identical to the path-dependent serial dictatorship defined there: \mathcal{E}^c is defined such that agent 1 chooses house g if and only if he ranks house d lowest. Consequently, for \mathcal{E}^c the given form of bossy serial dictatorship is ex-ante Pareto-dominated by the alternative path-dependent serial dictatorship γ' provided the same example. This is but one example, it is not known whether ex-ante Pareto optimality generally conflicts with bossiness.

8 Appendix

Proof of Lemma 1.

The assumption that

$$\sum_{E \in P} \pi(E) \max_{h \in S} \bar{\omega}_h^i(E) - c(P) \neq \sum_{E \in Q} \pi(E) \max_{h \in S} \bar{\omega}_h^i(E) - c(Q)$$

holds for any i , any two i -affordable partitions $P \neq Q$, and all subsets of houses $S \subset H$, implies that for any pair of an agent and a choice set, (i, S) , there exists a *unique* best learning choice P that maximizes agent i 's ex ante utility of his - informed - choice. Next the assumption that $\bar{\omega}_h^i(E) \neq \bar{\omega}_g^i(E)$ for any two houses $h \neq g$ and for any i -affordable event E , implies that agents are never indifferent between two houses for any ex post preferences. Together these assumptions imply that any $\langle \gamma, t^\gamma \rangle(\mathcal{E})$ has a unique equilibrium (and thereby also a unique outcome).¹⁰ \square

Proof of Lemma 2. Suppose $\psi^{c,b}$ is such that there is no σ, d, g with $(c_\sigma, b_\sigma)(d) = (i, ow)$, $(c_\sigma, b_\sigma)(g) = (j, ow)$ and $i \neq j$. For $\psi^{c,b}$ not to be a path-dependent serial dictatorship, there has to be at least one σ^b with a broker. Let σ° be a last matching following on σ^b with a broker. Let j be broker of d at σ° , formally $\sigma^b \subset \sigma^\circ$, $(c_{\sigma^\circ}, b_{\sigma^\circ})(d) = (j, br)$ and there does not exist a σ'' such that $\sigma^\circ \subset \sigma''$, $(c_{\sigma''}, b_{\sigma''})(h) = (i, br)$ for some i, h .

Define a set of functions $(c', b') = \{(c'_\sigma, b'_\sigma) : \bar{H}_\sigma \rightarrow \bar{I}_\sigma \times (ow, br)\}_{\sigma \in \bar{\mathcal{M}}}$ by $(c'_{\sigma^\circ}, b'_{\sigma^\circ})(h) = (i^*, ow)$ for all $h \in \bar{H}_{\sigma^\circ}$ and $i^* \in \bar{I}_{\sigma^\circ}$ the only owner at σ° according to (c, b) . Otherwise (c, b) and (c', b') are identical, that is $(c_\sigma, b_\sigma) = (c'_\sigma, b'_\sigma)$ for all $\sigma \neq \sigma^\circ$. Next observe that (c', b') satisfies conditions R1-R6

¹⁰Without the refinement of truth-telling, there might be multiple equilibria. To see this observe that the first dictator is indifferent between the announcement of any preferences that rank his choice out of the grand set H highest. However, even if we do not impose the refinement This is implied by the non-bossiness of serial dictatorship.

and is therefore a control rights structure.¹¹ Now to see that $\psi^{c',b'} = \psi^{c,b}$, observe that under (c, b) agent i^* can choose among all houses still available at σ° since the broker is forced to point to some house owned by i^* . On the other hand, if i^* does point to house d at σ° agent j 's choice set is that of all houses still available but d : $\overline{H}_{\sigma^\circ} \setminus \{d\}$. Agent j has the same choices under (c', b') since $(c_{\sigma^*}, b_{\sigma^*}) = (c'_{\sigma^*}, b'_{\sigma^*})$ for $\sigma^* = \sigma^\circ \cup \{(i^*, d)\}$. And by R6 we must have that $(c_{\sigma^*}, b_{\sigma^*})(h) = (j, ow)$ for all $h \in \overline{H}_{\sigma^*} = \overline{H}_{\sigma^\circ} \setminus \{d\}$. If σ° was the only σ with a broker, then we are done, since $\psi^{c',b'}$ is a path-dependent serial dictatorship. If not, define another control rights structure which eliminates brokerage from a submatching σ that is a last submatching with a broker according to (c', b') . Repeated application of this procedure terminates with a control rights structure (c^*, b^*) such that ψ^{c^*,b^*} is a path-dependent serial dictatorship and $\psi^{c^*,b^*} = \psi^{c,b}$. So any $\psi^{c,b}$ is either a path-dependent serial dictatorship or has two owners at some submatching σ . \square

Proof of Lemma 3.

Fix any $\psi^{c,b}$ such that there are two owners at some submatching σ . Let σ^* be the first in a sequence of such submatchings, formally $(c_{\sigma^*}, b_{\sigma^*})(k) = (j, ow)$ and $(c_{\sigma^*}, b_{\sigma^*})(g) = (j', ow)$ for some agents $j \neq j'$, houses k, g ; and $\sigma \subsetneq \sigma^*$ implies that there exists an agent i_σ that is the sole owner at σ . Assume w.l.o.g. that $j = 1, j' = 2$ and that the c-tree t is such that 1 has to announce his preferences before he learns the announcement of agent 2. Fix a submatching σ' such that $\sigma^* \subset \sigma'$ with $I_{\sigma'} = I \setminus \{1, 2\}$ and $H_{\sigma'} = H \setminus \{k, g\}$ and define the environment \mathcal{E}^A as follows:

Restricted to agents 1,2 and houses k, g , the environment \mathcal{E}^A is identical to the environment \mathcal{E}^a as defined in Example 4: for $i = 1, 2$: $\omega_k^i = 8$ and

¹¹R1 and R3 are satisfied since there are no more brokered houses under (c, b) than under (c', b') and since the brokers under (c', b') do control the same houses under (c, b) as under (c', b') . R2 is satisfied as the case of only one unmatched agent arises only for $|\sigma| = n - 1$ for housing problems as they are defined here and since (c, b) and (c', b') coincides for such submatchings. R4 is satisfied since (c', b') since the sole owner at σ° owns *more* houses under (c', b') than under (c, b) . R5 is vacuously satisfied since under (c', b') there is no submatching σ with two owners in \overline{I}_σ . R6 is satisfied since for the only pairs submatchings $\sigma \subset \sigma'$ with $|\sigma'| = |\sigma| + 1$ for which (c', b') differs from (c, b) the antecedent of R6 (that there should be two different agents controlling houses at σ) is not satisfied.

$\omega_k^i = 0$, each with probability $\frac{1}{2}$; $\omega_g^i = 2$, $c^1(\bar{P}^1) = .8$ and $c^2(\bar{P}^2) = 0$. Agents 1,2 prefer houses k, g to all other houses: $\omega_h^i < 0$ for all $h \in \bar{H}_\sigma \setminus \{k, g\}$. Finally, each remaining agent knows his preferences and strictly prefers his assignment under σ' to all other houses: $\omega_{\sigma'(i)}^i > \omega_h^i$ for all $h \neq \sigma'(i)$.

I claim that $\langle \varphi, t \rangle(\mathcal{E}^A)$ has a unique truth-telling equilibrium, this equilibrium results in the matching $\mu : I \rightarrow N$ with $\mu(i) = \sigma'(i)$ for $i \in I_{\sigma'}$, $\mu(1) = k$ and $\mu(2) = g$. All agents but 1 and 2 know their preferences. All these agents have a unique truth-telling strategy, which entails for agent i to point to house $\sigma'(i)$ as long as this house is still available. By assumption $\psi^{c,b}$ is a serial dictatorship until the submatching σ^* obtains. Since $\sigma^* \subset \sigma'$ the first I_{σ^*} dictators will choose houses in accordance with σ^* . At this point agent 1 becomes an owner of k and agent 2 becomes an owner of g . Given that $\omega_h^i < 0$ is known for all $h \in H_{\sigma'}$ for $i = 1, 2$, agents 1 and 2 point to houses k and g in any truth-telling equilibrium. Conversely, none of the remaining agents $\bar{I}_{\sigma^*} \setminus \{1, 2\}$ points to either house k or g . So agents 1 and 2 find themselves in the same choice-situation as they did in the housing problem constructed in Example 4, which implies that agent 1 appropriates k and agent 2 appropriates g in the unique equilibrium of $\langle \varphi, t \rangle(\mathcal{E}^A)$. Each of the remaining agents i points to $\sigma'(i)$ in the following rounds. Since $\sigma'(i) \neq \sigma'(i')$ for all $i \neq i'$, each agent $i \in I_{\sigma'}$ is matched to $\sigma'(i)$. To see that truth-telling is an equilibrium observe that each agent $i \in I \setminus \{1, 2\}$ obtains their most preferred house under the truth-telling strategy profile and can therefore not possibly be made better off through a deviation. So the profile of ex ante utilities implemented by $\langle \varphi, t \rangle(\mathcal{E}^A)$ is $(4, 2, \omega_{\sigma'(3)}^3, \dots, \omega_{\sigma'(n)}^n)$.

Now consider a serial dictatorship $\langle \delta, t^\delta \rangle$ with $[\delta]$ is the identity function. Since agents 1 and 2 prefer houses k and g to all other houses, their choices are as in the serial dictatorship discussed in Example 4. On the other hand, since each agent i strictly prefers his assignment under σ' to all other houses, all remaining agents will choose in accordance with σ' . So the unique profile of expected utilities implemented by the serial dictatorship $\langle \delta, t^\delta \rangle(\mathcal{E}^A)$ is $(4.2, 3, \omega_{\sigma'(3)}^3, \dots, \omega_{\sigma'(n)}^n)$, which Pareto-dominates the unique outcome of $\langle \psi^{c,b}, t \rangle(\mathcal{E}^A)$. \square

Proof of Lemma 4.

For γ not to be a simple serial dictatorship there have to be some submatchings $\bar{\sigma}$ and $\tilde{\sigma}$ such that $|I_{\bar{\sigma}}| = |I_{\tilde{\sigma}}|$ but $\gamma(\bar{\sigma}) \neq \gamma(\tilde{\sigma})$. Let $\bar{\sigma}$ and $\tilde{\sigma}$ be such pair of minimal length and assume without loss of generality that $\bar{\sigma}(i) \neq \tilde{\sigma}(i)$ for exactly one agent i . Assume that this agent is agent 1 and that $\bar{\sigma}(1) = g$, $\tilde{\sigma}(1) = k$, $\gamma(\bar{\sigma}) = 2$, and $\gamma(\tilde{\sigma}) = 3$.

Now consider a submatching σ' with $I_{\sigma'} = I \setminus \{1, 2, 3\}$ and $H_{\sigma'} = H \setminus \{g, k, d\}$. Define an environment \mathcal{E}^C , such that restricted to agents 1,2,3 and houses g, k , and d the environment is identical to \mathcal{E}^c , the environment presented in Example 5. Furthermore, assume that all agents in $I_{\sigma'}$ know their values of all houses, in particular they know that $\omega_{\sigma'(i)}^i > \omega_h^i$ holds for all $h \neq \sigma'(i)$. Finally, agents 1,2,3 know that $\omega_h^i < 0$ for any $i = 1, 2, 3$ and $h \notin \{g, k, d\}$.

In the unique equilibrium of $\langle \gamma, t \rangle(\mathcal{E}^C)$, all agents in $I_{\sigma'}$ are matched to σ' . The situation faced by agents 1,2 and 3 is identical to that they face in Example 5. Just as in that example, it would be a Pareto improvement for agents 2 and 3 to “switch”. So the alternative path-dependent serial dictatorship (γ', t') with $\gamma'(\sigma) = 2$ if $\gamma(\sigma) = 3$ and $\gamma'(\sigma) = 3$ if $\gamma(\sigma) = 2$ ex ante Pareto-dominates the given mechanism $\langle \gamma, t \rangle$ for the environment just constructed.

□

Proof of Lemma 5.

Consider a dynamic simple serial dictatorship $\langle \delta, t \rangle$ such that the c-tree t is not δ^t . Assume without loss of generality that δ is the identity function, so agents become dictators in the order of their names (one, two, three, ...). For t not to be δ^t there has to be some submatching σ such that some agent $i > |I_{\sigma}| + 1$ needs to announce his preferences after σ . Consider σ^* as the shortest submatching with this property. And let $i^* > |I_{\sigma^*}| + 1$ be such that t prescribes for i^* to announce his ranking at the submatching σ^* . Let $|I_{\sigma}| + 1 : i$. Observe that $i^*, i' < n$ as agent n need not announce any ranking in the simple serial dictatorship δ with δ the identity.

Consider a submatching σ' such that $\sigma^* \subset \sigma'$, $I_{\sigma'} = I \setminus \{i^*, i', n\}$ and $H_{\sigma'} = H \setminus \{g, k, d\}$. Now use this submatching to define an environment

\mathcal{E}^B as follows: For any $i \in I_{\sigma'}$, assume that he knows his preferences over all houses and that $\omega_{\sigma'(i)}^i > \omega_h^i$ holds for all $h \neq \sigma'(i)$. For agents i^*, i', n assume that $\omega_h^i < 0$ is known for all $h \in H_{\sigma'}$. Assume furthermore that restricted to houses $\{d, g, k\}$ and agents i^*, i', n the environment is identical to \mathcal{E}^b , the environment defined in Example 2, meaning each of these agents faces a small cost to learn his value of house d which has a high variance. Each of these two agents knows the value of houses g and k .

In the unique equilibrium of $\langle \delta, t \rangle(\mathcal{E}^B)$, all agents $i \in I_{\sigma'}$ obtain house $\sigma'(i)$. Moreover both agent i and i^* will spend to learn their value of house d and the allocation of houses g, k, d to agents i^*, i' and n is such as if they knew its value. To see that $\langle \delta, t \rangle$ is dominated by a 3S dictatorship at \mathcal{E}^B observe that unique equilibrium of the corresponding 3S dictatorship $\langle \delta, t^\delta \rangle(\mathcal{E}^B)$ differs from the unique equilibrium of $\langle \delta, t \rangle(\mathcal{E}^B)$, only in one respect: agent i^* will only learn his value of house d if this information is relevant to his decision, that is if agent i' does not himself acquire house d . The equilibrium allocation is identical for both cases. In sum, the ex-ante utility of all agents but agent i^* remains unchanged: the allocations of both equilibria, moreover agent i' acquires information under both equilibria. Agent i^* prefers the equilibrium of $\langle \delta, t^\delta \rangle(\mathcal{E}^B)$ since his expected cost of information acquisition is lower in this equilibrium. \square

Proof of Theorem 2.

This proof consists in a few amendments of the proof of Theorem 1. To prove the first part, I define a selection procedure among the outcomes $o((\delta, t^s)(\mathcal{E}))$ in case there are any indifferences. The problem of the first dictator in the game $(\delta, t^s)(\mathcal{E})$ is identical to his problem in the game $(\delta, t^\delta)(\mathcal{E})$. So he has a unique best strategy. However, the second dictator may not condition his learning on the first dictator's choice. He might have multiple best strategies. If he does so, choose the strategy most preferred by the third dictator; if that does not lead to a unique strategy, choose the one that is best according to the next dictator until one arrives at the last dictator. If this procedure does not lead to a unique best strategy, pick any strategy among the set of strategies for which all agents are indifferent. Now, conditioning on the second dictator's strategy, choose the third dictator's strategy in the

same fashion and so forth. This procedure leads to a subset of outcomes $o^*((\delta, t^s)(\mathcal{E})) \subset o((\delta, t^s)(\mathcal{E}))$. The proof of the part 1) of the present theorem consists in an application of the proof of part 1) of Theorem 1 to this subset of outcomes.

Next, for part 2), observe that, following Lemma 2, we can partition the set of strategy-proof and non-bossy $\langle \varphi, t^s \rangle$ that are not simultaneous simple serial dictatorships into two subcategories: $\langle \varphi, t^s \rangle$ with φ a path-dependent serial dictatorship and φ has a round with at least two owners. Now reconsider Lemma 3 and observe that the mechanism with at least two owners in some round φ together with the environment \mathcal{E}^A , and the 3S dictatorship $\langle \delta, t^\delta \rangle$ constructed in the proof of that Lemma are such that $o(\langle \delta, t^\delta \rangle(\mathcal{E}^A)) = o((\delta, t^s)(\mathcal{E}^A))$. The reason for this is that according to the equilibrium strategies in $\langle \delta, t^\delta \rangle(\mathcal{E}^A)$ only one agent learns anything, all other agents announce their ex-ante preferences. Therefore it does not matter whether all agents are forced to announce simultaneously. So the construction in the proof of Lemma 3 also serves to show that for any $\langle \varphi, t^s \rangle$ with φ a trading cycles mechanism with at least two owners in some round there exists a simple serial dictatorship such that $\langle \delta, t^s \rangle(\mathcal{E}^A) \succ^* \langle \varphi, t^s \rangle(\mathcal{E}^A)$.

Next, observe that the environment used to prove Lemma 4 has only one agent with a priori unknown preferences. Therefore within that environment the choice of a c-tree never matters $\langle \varphi, t \rangle(\mathcal{E}^C) = \langle \varphi, t^s \rangle(\mathcal{E}^C)$. Therefore the conclusion of Lemma 4 remains valid when restricting attention to simultaneous mechanisms. It can be concluded for the two relevant cases an environment together with a Pareto-dominating path-dependent serial dictatorship is available. \square

References

- [1] Abdulkadiroglu, A. and T. Sonmez: “Random Serial Dictatorship and the Core from Random Endowments in House Allocation Problems”, *Econometrica*, 66, (1998), pp. 689 - 701.
- [2] Bergemann D. and J. Valimaki: “Information in Mechanism Design”, in Whitney Newey Richard Blundell and Torsten Persson, editors, *Proceed-*

- ings of the 9th World Congress of the Econometric Society*, pp. 186 - 221. Cambridge University Press, 2007.
- [3] Compte, O. and P. Jehiel: “Information acquisition in Auctions: Sealed-bid or Dynamic Formats?” *Rand Journal of Economics*, 38, (2007), pp. 355 - 372.
 - [4] Ehlers, L.: “Coalitional Strategy-Proof House Allocation,” *Journal of Economic Theory*, 105, (2002), pp. 298 - 317.
 - [5] Gerardi, D. and L. Yariv: “Information Acquisition in Committees”, *Games and Economic Behavior*, 62, (2008), pp. 436 - 459.
 - [6] Gershkov A. and B. Szentes: “Optimal Voting Schemes with Costly Information Aquisition”, *Journal of Economic Theory*, 144, (2009), pp. 36 - 68.
 - [7] Papai, S.: “Strategyproof Assignment by Hierarchical Exchange”, *Econometrica*, 68, (2000), pp. 1403 - 1433.
 - [8] Pycia, M. and Unver, U.: “A Theory of House Allocation and Exchange Mechanisms,” *mimeo* UCLA, January 2009.
 - [9] Satterthwaite M. and H. Sonnenschein: “Strategy-Proof Allocation Mechanisms at Differentiable Points,” *Review of Economic Studies*, 48, (1981), pp. 587 - 597.
 - [10] Shapley, L. and H. Scarf: “On Cores and Indivisibility”, *Journal of Mathematical Economics*, 1, (1974), pp. 23 - 37.
 - [11] Smorodinsky R. and M. Tennenholtz: “Overcoming Free Riding in Multi-Party Computations - The Anonymous Case”, *Games and Economic Behavior*, 55, (2006), pp. 385 - 406.
 - [12] Takamiya, K.: “Coalition Strategy-Proofness and Monotonicity in Shapley-Scarf Housing Markets,” *Mathematical Social Sciences*, 41, (2001), pp. 201 - 213.