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A Contribution to the Theory of Optimal Utilitarian Income Taxation*

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Abstract

The paper provides a new formulation of the Mirrlees-Seade theorem on the positivity of the optimal marginal income tax, under weaker assumptions and in a more general model. The formulation of the theorem is independent of whether the model involves finitely many types or a continuous type distribution. The formal argument makes the underlying logic transparent, relating the mathematics to the economics and showing precisely how each assumption enters the analysis.

Key Words: Optimal Income Taxation, Utilitarian Welfare Maximization, Redistribution

JEL Classification: D63, H21

1 Introduction

The theory of optimal utilitarian income taxation is one of the cornerstones of normative public economics. Its central result states that, under certain assumptions, when individual productivity (earning ability) is a hidden characteristic, the optimal marginal income tax is strictly positive at all income levels other than the very top and, in some specifications, the very bottom of the income distribution. The first version of this result was established by

*This is a revision of Hellwig (2005), which extends the analysis of the earlier paper to allow efficient outcomes involving zero production of people with low productivity. I am very grateful to Felix Bierbrauer, Christoph Engel, Thomas Gaube, Stefan Homburg, Peter Norman, the Editor and two referees for very helpful discussions and comments on earlier drafts. The usual disclaimer applies.

Mirrlees (1971, 1976) for a model with a continuum of types. His result was subsequently generalized by Seade (1977, 1982), Ebert (1992), and Brunner (1993, 1995). For models with finitely many types, different versions have been provided by Guesnerie and Seade (1982), Stiglitz (1982), Roëll (1985), and Weymark (1986).

It is not clear whether this cornerstone of public economics is really a monolith or whether it is a couple of stones packaged under a common label. Models of optimal income taxation with finitely many types and with a continuum of types involve different assumptions and different formal arguments. Therefore, it is difficult to see the underlying common principle.

Our understanding of underlying principles is particularly deficient in models with a continuum of types. The utility specification that is used in these models is special, the role of the assumptions is difficult to understand, and the relation of the mathematical arguments to the underlying economics is unclear. The utility specification identifies the hidden characteristic with the wage rate, i.e., the ratio of output produced or income earned to hours worked. This is problematic because, in many settings, the wage rate can be taken to be observable.

In these models, the utility function is assumed to be strictly concave and leisure is assumed to be a normal good. Together, these two conditions imply that some redistribution is desirable. However, normality of leisure is an *ordinal* property of the utility function. Why this ordinal property should matter for redistribution is unclear.

In these models we also use formal arguments which mix local and global considerations in ways that are difficult to disentangle. The arguments actually depend on whether consumption and leisure are (Edgeworth) substitutes or complements. This makes it hard to see the common underlying structure and to relate it to any economic intuition.

The present paper develops a unified approach to optimal utilitarian income taxation. Relying on a new proof strategy, it shows that the theory of optimal income taxation is indeed a monolith, involving the same assumptions and the same arguments in models with finitely many types and in models with a continuum of types. The new approach provides for a sharper formulation of the theorem, in a more general model, under weaker assumptions. More importantly, it provides for a transparent account of the underlying logic, relating the mathematics to the economics and making clear where exactly and how the key assumptions enter the analysis. The argument is the same for models with finitely many types and with a continuum of types. For models with a continuum of types, I abandon the interpretation of the hidden characteristic as a wage rate and use the abstract "type" formulation of Guesnerie and Seade (1982); moreover, there is no need to have different arguments when consumption and leisure are substitutes and when they are complements.

The argument proceeds indirectly. Exploiting an idea of Matthews and Moore (1987), I study the modification of the optimal income tax problem that is obtained if upward incentive constraints are replaced by a monotonicity condition on consumption. Under fairly general assumptions, this modified problem turns out to be equivalent to the optimal income tax problem in that any solution to one problem is also a solution to the other and *vice versa*. However, the modified problem is much easier to analyse.

In the modified problem, it is easy to see that optimal marginal income taxes are never negative. Negative marginal income taxes, i.e., a subsidization of work at the margin, would induce people to work more and to consume more than is efficient.[1] Such a distortion could only be justified by *upward* incentive constraints requiring excessive workloads as a way of preventing people with lower earning abilities from imitating people with higher earning abilities. When upward incentive constraints are not imposed, there cannot be any rationale for such distortions. Because one does not have to worry about negative marginal taxes, it is easier to see which conditions call for marginal taxes to be *strictly positive*.

The rationale for utilitarian redistribution will be based on a version of Roëll's (1985) "very weak redistribution assumption". It postulates that, in the absence of incentive considerations, certain allocations cannot be welfare maximizing because utilitarian welfare can be increased by redistribution from people with higher productivity to people with lower productivity, requiring the former to consume less or to work more so that the latter can consume more or work less. Whereas Roëll imposed this postulate for all incentive-compatible allocations, I only impose it for allocations in which the outcomes for the less productive people are efficient and the outcomes for the more productive people are either efficient or distorted in the direction

[1] This argument assumes that, as in Mirrlees (1971), consumption-leisure choices are made at the intensive margin. Saez (2002) has shown that negative marginal income taxes can be optimal if consumption-leisure choices are made at the extensive margin, so people of type n choose only whether to work at a job of type n or not to work; see also Laroque (2005). In this case, there is no question of a person of type n choosing to work at a job of type n-1 and earning a lower income; marginal income taxes as such are therefore economically unimportant. The optimal tax on income earned on a job of type n is obtained from a simple elasticities consideration involving only people of type n. A model involving choices at both the intensive and the extensive margin would have to allow for multidimensional type heterogeneity; for a first approach to this problem, see Choné and Laroque (2006).

of too little consumption and too much leisure.

Under this redistribution assumption, the paper shows that the optimal income tax schedule is strictly increasing. If the type set is finite, the optimal marginal income tax is strictly positive at all positive income levels below the top. If the type set is an interval and the distribution of types has a continuous and strictly positive density on this interval, the optimal marginal income tax is strictly positive. Indeed, on any compact set of income levels between the bottom and the top of the income distribution, it is bounded away from zero. There is not even a possibility that the optimal marginal income tax might be zero or arbitrarily close to zero near some isolated point.[2]

These results can be understood in terms of a local equity-efficiency tradeoff. If the marginal tax rate that is relevant for a given type is zero, then this type's consumption-leisure choice is efficient. Given the information that marginal tax rates are never negative, any higher type's consumption-leisure choice is either efficient or distorted in the direction of too little consumption and too much leisure. The assumption on the desirability of redistribution therefore implies that, in this case, it is desirable to have some redistribution from people with the higher types to people with the lower type. *Ceteris paribus*, such a redistribution may violate downward incentive compatibility, but, by standard arguments, the incentive effects can be neutralized by distorting the lower type's consumption-leisure choice in the direction of too little consumption and too much leisure. If this shift is small, then, because one starts from an efficient pair, the efficiency loss that the shift induces is negligible and is outweighed by the gain from the redistribution.

In this analysis, the rationale for utilitarian redistribution rests on both the cardinal *and* ordinal properties of the utility specification. The utilitarian approach to income taxation has traditionally built on the notion that it is desirable to redistribute consumption because people with higher consumption have a lower marginal utility of consumption. In the theory of income taxation, this notion goes back (at least) to Edgeworth (1897/1925).[3] Mirrlees (1971, 1976), as well as Guesnerie and Seade (1982) and Weymark (1986), have followed this tradition. They recognized that differences in

[2] By contrast, Seade (1982) proceeds by assuming that positivity fails on some interval and then obtains a contradiction by comparing the consumption-income pairs that are associated with the two endpoints of the interval. This argument neglects the possibility that the interval in question might be degenerate, i.e., consist of a single point, in which case the comparison in question is moot.

[3] For a comprehensive account, see Chapter 5 of Musgrave (1959).

consumption levels which are due to differences in the hidden productivity parameters are likely to be correlated with differences in leisure, but, through additional assumptions, they avoided the complications that this might cause. Thus, in Mirrlees (1971), additive separability of utility functions eliminates the possibility that people who produce more might also be hungrier.

The focus on redistributing consumption was loosened by Seade (1982) and Roëll (1985). For Mirrlees's (1971) model, Seade (1982) showed that the assumption of additive separability of the utility function can be replaced by the assumption that leisure is a noninferior good. For the more abstract utility specification of Guesnerie and Seade (1982), Roëll (1985) showed that the traditional redistribution assumption in terms of consumption can be weakened to the assumption that it is desirable to redistribute consumption *or* leisure from participants with higher roductivity to participants with lower productivity.[4] She also showed that, for the utility specification of Mirrlees (1971), this condition is implied by noninferiority of leisure.[5]

This paper goes one step further and all but eliminates the notion that it is desirable to redistribute consumption. Desirability of redistribution, as formulated in this paper, refers on a redistribution of consumption *only* if the lower type is not working so that there is no scope at all for increasing this type's leisure. If the lower type is working, desirability of redistribution can be formulated solely in terms of leisure, having the high-productivity types work more so that the low-productivity types can have more leisure.

At this point, it becomes difficult to distinguish whether the desirability of redistribution is based on inequality aversion or on the consideration that the aggregate burden of producing a given aggregate output is reduced if output requirements are redistributed from people with low productivity to people with high productivity. I will actually show that, whatever the ordinal properties of the utility function may be, the redistribution assumption on which I rely holds whenever inequality aversion is sufficiently large. Normality of leisure is not necessary. However, I will also give an example to show that the redistiribution assumption can hold even though inequality aversion is zero.

In the following, Section 2 formulates the optimal income tax problem. Section 3 states and discusses the assumptions that I impose. Section 4 explains the main argument. Section 5 provides the formal analysis for the

[4] In the formal statement of her theorem, Roëll only claims nonnegativity of the optimal marginal income tax. However, the arguments given by Guesnerie and Seade (1982) for strict positivity apply to her specification as well.

[5] On this point see also Brunner (1993, 1995) and Homburg (2004).

case of finitely many types, Section 6 for the case when the type set is an interval and the type distribution has a continuous, strictly positive density. Some supplementary proofs are given in the Appendices.

2 The Optimal Income Tax Problem

Following Mirrlees (1971, 1976) and Seade (1977, 1982), I study a large economy with one produced good and labour. Each agent in the economy is characterized by a productivity parameter n. An agent with productivity parameter n who consumes c units of the produced good and who supplies the labour needed to produce y units of output obtains the payoff $u(c, y, n)$. The leading example in the literature is the specification

$$u(c, y, n) = U(c, \frac{y}{n}). \tag{2.1}$$

In this specification, n is labour productivity (the wage rate) and $\frac{y}{n}$ is the number of hours the person needs to work to produce the output y or to obtain the labour income y. The analysis here encompasses (2.1), but is not limited to this specification.

The productivity parameter n of any one person is the realization of a nondegenerate random variable $\tilde{n}$ with probability distribution F which has a compact support $N \subset \Re_+$. The smallest and largest elements of N are denoted as n^0 and n^1. The distribution F is the same for all agents. By a large-numbers effect, F is also assumed to be the cross-section distribution of the realizations of people's productivity parameters.

In this economy, an *allocation* is a pair of functions, $(c(\cdot), y(\cdot))$, which indicate how an individual's consumption level $c(n)$ and output provision level $y(n)$ depend on his productivity parameter n. An allocation is *feasible* if

$$\int_N c(n)dF(n) \leq \int_N y(n)dF(n), \tag{2.2}$$

so aggregate consumption does not exceed aggregate production. The allocation is *incentive-compatible* if

$$u(c(n'), y(n'), n') \geq u(c(n), y(n), n') \tag{2.3}$$

for all n and n' in N, so nobody has an incentive to claim that his productivity parameter is n when in fact it is n'. An individual's productivity parameter and labour input are taken to be private information, so incentive compatibility is a prerequisite for the implementation of an allocation.

Allocations are assessed according to the utilitarian welfare functional

$$\int_N u(c(n), y(n), n) dF(n), \tag{2.4}$$

The utilitarian welfare maximization problem is to maximize (2.4) over the set of feasible and incentive-compatible allocations. By the *taxation principle* of Hammond (1979) and Guesnerie (1995), this problem is equivalent to the problem of choosing an optimal tax schedule $T(\cdot)$ and then letting each person choose an output provision level y and a consumption level $c = y - T(y)$. Therefore, I refer to it as the *optimal income tax problem.*

I shall be interested in the efficiency properties of optimal allocations. For any n and any v, let $(c^*(n, v), y^*(n, v))$ be the pair which provides the person with productivity parameter n with the utility v at the lowest cost in terms of required net resources, i.e., let $(c^*(n, v), y^*(n, v))$ be the solution to the problem

$$\min_{u(c,y,n) \geq v} (c - y). \tag{2.5}$$

A consumption/output pair $(c(n), y(n))$ providing a person of type n with the utility

$$v(n) = u(c(n), y(n), n) \tag{2.6}$$

is said to be *efficient* for n if

$$(c(n), y(n)) = (c^*(n, v(n)), y^*(n, v(n))); \tag{2.7}$$

the pair $(c(n), y(n))$ is said to be *distorted downward from efficiency* if

$$(c(n), y(n)) \ll (c^*(n, v(n)), y^*(n, v(n))) \tag{2.8}$$

and to be *distorted upward from efficiency* if

$$(c(n), y(n)) \gg (c^*(n, v(n)), y^*(n, v(n))). \tag{2.9}$$

If the utility function is differentiable, efficiency implies that

$$u_c(c(n), y(n), n) + u_y(c(n), y(n), n) \leq 0, \tag{2.10}$$

with a strict inequality only if $c(n) = 0$ or $y(n) = 0$; if $(c(n), y(n))$ is distorted downward from efficiency, then

$$u_c(c(n), y(n), n) + u_y(c(n), y(n), n) > 0. \tag{2.11}$$

3 Assumptions

The following assumptions will be imposed throughout the paper.

RC **Regularity Conditions:** The utility function $u : \Re^3_+ \to \Re$ is twice continuously differentiable as well as increasing in c, decreasing in y, nondecreasing in n, and strictly quasiconcave in c and y. Moreover, for every $n \in N$ and every v in the range of $u(\cdot, \cdot, n)$, the efficient pair $(c^*(n,v), y^*(n,v))$ is well defined.

ND **Nondegeneracy:** For any n, let

$$c^{LF}(n) = y^{LF}(n) = \arg\max_y u(y, y, n) \tag{3.1}$$

be the outcome that a person of type n would choose under *laissez-faire.* Then

$$(c^{LF}(n^1), y^{LF}(n^1)) \gg (0, 0).$$

SSCC **Strict Single-Crossing Condition:** The utility function satisfies

$$\frac{\partial}{\partial n}\left[\frac{u_c(c,y,n)}{|u_y(c,y,n)|}\right] > 0 \tag{3.2}$$

for all $(c, y, n) \in \Re^2_{++} \times (0, 1)$.

DR **Desirability of Redistribution:** For any $(c, y, n) \in \Re^2_+ \times [n^0, n^1)$, there exists $\varepsilon > 0$ such that $n + \varepsilon \in N$, and, for all $n' \in (n, n + \varepsilon]$ and all $(c', y') \in \Re^2_+$, the following hold:

(a) if $c' = c$ and $y' < y$, then

$$|u_y(c,y,n)| > \left|u_y(c', y', n')\right|; \tag{3.3}$$

(b) if $c' > c$, $u(c', y', n') \geq u(c, y, n')$, and if, moreover (c, y) is efficient for n, and (c', y') is efficient or distorted downwards from efficiency for n', then

$$|u_y(c,y,n)| > \min(u_c(c', y', n'), \left|u_y(c', y', n')\right|) \;\text{ if }\; y > 0, \tag{3.4}$$

and

$$u_c(c,y,n) > \min(u_c(c', y', n'), \left|u_y(c', y', n')\right|) \;\text{ if }\; y = 0. \tag{3.5}$$

Conditions RC, ND, and SSCC are standard. In RC, I postulate quasi-concavity rather than concavity of u. Concavity can contribute to making redistribution desirable, but, as indicated by Proposition 2.1 below, this appears in the context of condition DR. ND rules out the trivial case where nobody produces or consumes anything. SSCC reflects the notion that the tradeoff between consumption and leisure results in a higher level of consumption and a lower level of leisure (more output provision) when the productivity parameter is higher.[6]

Condition DR provides the rationale for utilitarian redistribution. The condition implies that, if incentive considerations could be neglected, then, at certain specified allocations, it would be desirable to have some redistribution from people with a higher productivity parameter to people with a lower productivity parameter, requiring the former to work more or to consume less so that the latter can consume more or work less. For any allocation that satisfies $c(n') = c(n)$ and $y(n') < y(n)$ for given n and n' slightly greater than n, part (a) of DR implies that welfare is increased if output requirements are redistributed from people with productivity parameter n' to people with productivity parameter n. For any allocation that satisfies $c(n') \geq c(n)$ and $u(c(n'), y(n'), n') \geq u(c(n), y(n), n')$ for $n' > n$, part (b) of DR asserts that, if $(c(n), y(n))$ is efficient for n and $(c(n'), y(n'))$ is efficient or distorted downward from efficiency for n', then welfare is increased if people with productivity parameter n' are required to consume less or to work more and if the workload of people with productivity parameter n is reduced if this workload is positive; if this workload is zero, welfare is increased by raising their consumption.

Part (a) of condition DR has no explicit analogue in the literature. If u is concave in c and y, this condition corresponds to the requirement that the marginal disutility of output provision, $|u_y(c, y, n)|$, be decreasing in n. It is automatically fulfilled if the utility function takes the form (2.1) for some concave U. For arbitrary utility functions, this is not necessarily the case. By stating it as an explicit assumption, I eliminate the possibility that, when upward incentive constraints are neglected, it might be desirable to choose an allocation with $c(n) = c(n')$ and $y(n) > y(n')$ for some n and $n' > n$.

Part (b) of condition DR can be read as a version of the "very weak redistribution assumption" that Roëll (1985) formulated for the finite-type

[6] Actually, one only needs the strictness of the inequality in (3.2) at n^1, the top of the type set; below the top, it is enough to have a weak inequality in (3.2).

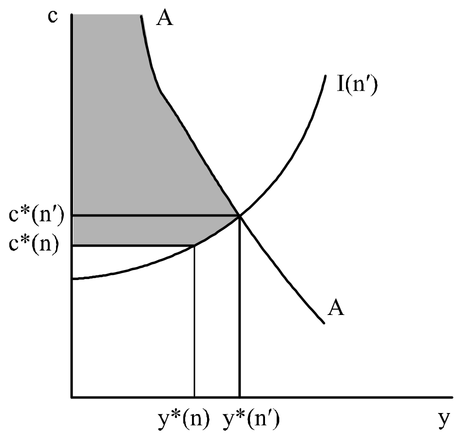

Figure 1: The set of outcome pairs (c′, y′) for n′ to which part (b) of DR applies

model of Guesnerie and Seade (1982). Roëll's condition is equivalent to the requirement that, for any two neighbouring types n and $n' > n$ with outcome pairs (c, y) and (c', y') that satisfy $c' \geq c$, and $u(c', y', n') \geq u(c, y, n')$, the sum of utilities is increased if a person of type n' is made to work more or to consume less in order that a person of type n can consume more or work less. This is also what the inequalities (3.4) and (3.5) are saying, (3.4) for the case $y > 0$, (3.5) for the case $y = 0$.

By contrast to Roëll (1985), I do not require these inequalities to hold for all incentive-compatible outcome pairs (c, y) and (c', y'). I only require them to hold for downward incentive-compatible outcome pairs (c, y) and (c', y') with $c' \geq c$, such that (c, y) is efficient for n and (c', y') is efficient or distorted downwards from efficiency for n'. An illustration is given in Figure 1. In this figure, $(c^*(n), y^*(n))$ is an efficient outcome pair for type n, $I(n')$ is the indifference curve of type $n' > n$ through $(c^*(n), y^*(n))$, and $A - A$ is the locus of efficient outcome pairs for type n'. Given that, in the figure, $y^*(n) > 0$, part (b) of DR postulates that, if $n' > n$ is sufficently close to n, then (3.4) holds for all outcome pairs (c', y') in the shaded area to the left of the locus $A - A$ of efficient outcome pairs for type n', above the indifference curve $I(n')$, and strictly above the horizontal line through the reference pair $(c^*(n), y^*(n))$.

Whenever y and y' are both positive, desirability of redistribution can be formulated in terms of output requirements rather than consumption. Under the premises of part (b) of condition DR, $y' > 0$ implies $u_c(c', y', n') \geq$

$|u_y(c', y', n')|$. Condition (3.4) can then be written as:

$$|u_y(c, y, n)| > |u_y(c', y', n')| \quad \text{if} \quad y > 0 \text{ and } y' > 0. \tag{3.6}$$

By contrast, redistribution of consumption plays a role if y and y' are both zero and the outcome pairs (c, y) and (c', y') are both efficient.

What does condition DR mean for the utility function u? For utility functions taking the special form (2.1), Roëll (1985) has shown that her "very weak redistribution assumption" is implied by strict concavity of u in combination with Seade's (1982) assumption that leisure is a noninferior good.[7] The following result provides a more general set of sufficient conditions under which u satisfies DR. These conditions do not presume that u takes the special form (2.1) or that leisure is a noninferior good.

Proposition 3.1 *Assume RC, ND, and SSCC, and suppose that, for any n, the indifference curves of the utility function $u(\cdot, \cdot, n)$ have strictly positive Gaussian curvature.*[8] *Assume also that N is an interval. Then condition DR holds if u is concave in c and y and, moreover,*

$$u_{cn}(c, 0, n) \leq 0, \quad u_{cc}(c, 0, n) < 0, \tag{3.7}$$

$$u_{yn}(c, y, n) > 0 \tag{3.8}$$

for all (c, y, n), and

$$u_{ny}(c^*(n, v), y^*(n, v), n)\frac{\partial y^*}{\partial v} + u_{nc}(c^*(n, v), y^*(n, v), n)\frac{\partial c^*}{\partial v} < 0, \tag{3.9}$$

for all $n \in N$ and all v in the range of $u(\cdot, \cdot, n)$. Under these assumptions on the functions u, c^*, and y^*, condition DR also holds if N is a finite set and the differences between neighbouring elements of N are uniformly small.

For a proof of Proposition 3.1, the reader is referred to Appendix A. The proposition reduces the desirability of redistribution to *four substantive properties of the utility function*: First, u is concave. Second, if $y = 0$, a person with higher consumption and higher productivity has a lower marginal utility of consumption. Third, the marginal disutility of producing additional output is decreasing in n. Fourth, for any $n' \in N$, the function $v \to u_n(c^*(n', v), y^*(n', v), n')$ is differentiable, and its slope is negative.

[7] See also Brunner (1993, 1995), and Homburg (2004).

[8] I.e., that the quadratic form $u_y^2 u_{cc} - 2u_c u_y u_{cy} + u_c^2 u_{yy}$ is everywhere strictly negative.

Thus, in Figure 1, u_n is *strictly decreasing* as one is moving up along the locus $A - A$ of efficient outcome pairs for type n'.

The first three properties are familiar from the literature. Concavity of u reflects inequality aversion. Monotonicity of $u_c(c, 0, n)$ reflects the idea that people with higher productivity are no better than people with lower productivity for transforming consumption into utility. Positivity of u_{yn} reflects the idea that people with higher productivity have a lower marginal disutility from producing additional output.

As for condition (3.9), observe that the set of outcome pairs that are efficient for n' (the line A - A in Figure 1) coincides with the set of solutions to the problem of maximizing $u(c, y, n')$ under the budget constraint $c = \alpha + y$, for α varying over the real numbers. For given α and n', let

$$v^*(\alpha, n') := \max_{y \geq 0} u(\alpha + y, y, n') \tag{3.10}$$

be the utility maximum in this problem. Under the given assumptions, the indirect utililty function v^* is twice continuously differentiable, with $v^*_n = u_n$ and

$$v^*_{n\alpha} = \left[u_{ny}(c^*(n,v), y^*(n,v), n) \frac{\partial y^*}{\partial v} + u_{nc}(c^*(n,v), y^*(n,v), n) \frac{\partial c^*}{\partial v} \right] v^*_\alpha. \tag{3.11}$$

Thus, condition (3.9) is equivalent to the requirement that $v^*_{n\alpha}(\alpha, n') < 0$ for all n' and α. This inequality in turn is equivalent to the requirement that $v^*_{\alpha n}(\alpha, n') < 0$, i.e., that the "social marginal utility of income" in the absence of distortionary taxation is a decreasing function of n. This latter condition figures prominently in the literature on optimal *linear* income taxation, see, e.g., Hellwig (1986).

For utility functions taking the form (2.1), the inequalities $v^*_{\alpha n} < 0$ and, therefore, (3.9) are implied by strict concavity of u and noninferiority of leisure.[9] For arbitrary utility functions, (3.9) holds if consumption is normal,[10] leisure is noninferior, and if $u_{yn} > 0$ and $u_{cn} < 0$.

Regardless of the signs of $\frac{\partial y^*}{\partial v}$ and $\frac{\partial c^*}{\partial v}$, i.e. *regardless of the ordinal* properties of u, condition DR is *always* satisfied if u is sufficiently concave. To see this, suppose that $u = \varphi \circ \hat{u}$, where $\hat{u}$ satisfies RC, ND, and SSCC, and φ is twice continuously differentiable, increasing, and strictly concave.

[9] See Christiansen (1983) or Werning (2000). For such utility functions, Roy's identity implies that the indirect utility function v^* satisfies $v^*_n = v^*_\alpha \frac{y}{n}$. Hence, $v^*_{\alpha n} = v^*_{n\alpha} = v^*_{\alpha\alpha} \frac{y}{n} + v^*_\alpha \frac{y_\alpha}{n}$.

[10] Under RC and SSCC, (3.9) is also satisfied if consumption is nonnormal and $u_{yn} > 0$.

Then u and $\hat{u}$ have the same ordinal properties. The elements of the Hessian of u are given as

$$u_{ij} = \varphi' \hat{u}_{ij} + \varphi'' \hat{u}_i \hat{u}_j$$

for $i, j \in \{c, y, n\}$. Thus, conditions (3.7) - (3.9) take the form

$$\varphi' \hat{u}_{cn} + \varphi'' \hat{u}_c \hat{u}_n < 0 \tag{3.12}$$

$$\varphi' \hat{u}_{yn} + \varphi'' \hat{u}_y \hat{u}_n > 0 \tag{3.13}$$

and

$$\varphi' \left[\hat{u}_{ny} \frac{\partial y^*}{\partial v} + \hat{u}_{nc} \frac{\partial c^*}{\partial v} \right] + \varphi'' \hat{u}_n < 0. \tag{3.14}$$

One easily verifies that these inequalities are fullied and, moreover, u is concave, if the curvature of φ is sufficiently large. This observation yields:

Corollary 3.2 *Let $\hat{u}$ be a utility function that satisfies RC, ND, and SSCC, and assume that, for any n, the indifference curves of $\hat{u}(\cdot, \cdot, n)$ have strictly positive Gaussian curvature. Let φ be twice continuously differentiable, increasing and strictly concave. If N is an interval, the function $u = \varphi \circ \hat{u}$ satisfies condition DR if the curvature $-\frac{\varphi''}{\varphi'}$ of the function φ is everywhere sufficiently large.*

Corollary 3.2 highlights the importance of the *cardinal* properties of u for the desirability of redistribution. Whereas Seade (1982) used strict concavity, a cardinal property, and noninferiority of leisure, an ordinal property, to derive the desirability of redistribution, Corollary 3.2 shows that, whatever the ordinal properties of u may be, some (local) redistribution of leisure is *always* desirable if $u(\cdot, \cdot, n)$ is sufficiently concave.

This being said, one should also see that, for reasons related to the *ordinal* properties of u, condition DR can be satisfied even if u is *not* strictly concave. For instance, the Cobb-Douglas specification

$$u(c, y, n) = c^{\beta} (1 - \frac{y}{n})^{1-\beta} \tag{3.15}$$

satisfies DR because the disutility of producing additional output is lower for people with higher n. With $v^*(\alpha, n) = \beta^{\beta}(1-\beta)^{1-\beta}(\alpha + n)n^{-(1-\beta)}$, inequality aversion does not play any role.

In the remainder of the paper, conditions RC, ND, SSCC, and DR will be imposed without further mention.

4 Standard Properties of Optimal Allocations

Past work on the optimal income tax problem has focussed on the following properties of optimal allocations.

Property A There is no distortion at the top: If $F(\{n^1\}) > 0$, then

$$(c(n^1), y(n^1)) = (c^*(n^1, v(n^1)), y^*(n^1, v(n^1))); \tag{4.1}$$

if $F(\{n^1\}) = 0$, then, for any sequence $\{n^k\}$ in N that converges to n^1 from below,

$$\lim_{k\to\infty} (c(n^k), y(n^k)) = (c^*(n^1, v(n^1)), y^*(n^1, v(n^1))). \tag{4.2}$$

Property B There are downward distortions between the top and the bottom: At any $n < n^1$ for which $F([n^0, n]) > 0$, $y(n) > 0$ implies

$$(c(n), y(n)) \ll (c^*(n, v(n)), y^*(n, v(n))). \tag{4.3}$$

Property C If $F(\{n^0\}) = 0$ and if $y(\cdot)$ is strictly increasing at $n = n^0$, there is no distortion at the bottom, i.e., for any sequence $\{n^k\}$ in N that converges to n^0 from above,

$$\lim_{k\to\infty} (c(n^k), y(n^k)) = (c^*(n^0, v(n^0)), y^*(n^0, v(n^0))). \tag{4.4}$$

Property D The functions $y(\cdot), c(\cdot)$, and $y(\cdot) - c(\cdot)$ are nondecreasing and co-monotonic on N, i.e., for any n and $n' > n$ in N, $y(n') \geq y(n)$, and,

$$\text{if } y(n') > y(n), \quad \text{then} \quad c(n') > c(n) \text{ and } y(n') - c(n') > y(n) - c(n), \tag{4.5}$$

$$\text{if } y(n') = y(n), \quad \text{then} \quad c(n') = c(n) \text{ and } y(n') - c(n') = y(n) - c(n). \tag{4.6}$$

Moreover, for some $\hat{n} \in (n^0, n^1)$, one has

$$c(n) > y(n) \quad \text{if} \quad n < \hat{n}, \tag{4.7}$$

and

$$c(n) < y(n) \quad \text{if} \quad n > \hat{n}. \tag{4.8}$$

Finally, $(c(n), y(n)) \ll (c^*(n^1, v(n^1)), y^*(n^1, v(n^1)))$ for all $n < n^1$.

The following result shows that, if Properties A - D hold, the optimal marginal income tax is zero at the top and strictly positive between the top and the bottom. This explains the importance of these properties for the theory of optimal income taxation.

Theorem 4.1 *If an allocation $(c(\cdot), y(\cdot))$ with associated indirect utility function $v(\cdot)$ exhibits Properties A - D, a tax schedule T that implements the allocation $(c(\cdot), y(\cdot))$ is strictly increasing on the range $y(N)$ of the output provision function. If T is differentiable,*[11] *its derivative $\tau(\cdot)$ satisfies the following:*

A: If $F(\{n^1\}) > 0$, then

$$\tau(y(n^1)) = 0; \tag{4.9}$$

if $F(\{n^1\}) = 0$, then, for any sequence $\{n^k\}$ in N that converges to n^1 from below,

$$\lim_{k\to\infty} \tau(y(n^k)) = 0. \tag{4.10}$$

B: For any $n \in [n^0, n^1)$, $F([n^0, n]) > 0$ and $y(n) > 0$ imply

$$\tau(y(n)) \in (0, 1). \tag{4.11}$$

C: If $F(\{n^0\}) = 0$ and if $y(\cdot)$ is strictly increasing at $n = n^0$, then, for any sequence $\{n^k\}$ in N that converges to n^0 from above,

$$\lim_{k\to\infty} \tau(y(n^k)) = 0. \tag{4.12}$$

Proof. Any tax schedule that implements $(c(\cdot), y(\cdot))$ satisfies

$$T(y(n)) = y(n) - c(n) \tag{4.13}$$

[11] If N is an interval and the allocation $(c(\cdot), y(\cdot))$ is continuous and strictly increasing, the tax schedule $T(\cdot)$ is necessarily differentiable on $y(N)$; its derivative $\tau(\cdot)$ is then given by (4.15). If $(c(\cdot), y(\cdot))$ is not strictly increasing, the tax schedule $T(\cdot)$ is not differentiable at any point at which there is bunching; in this case, if N is an interval and the allocation is continuous, the claims of the Theorem still hold for right-hand and left-hand derivatives of the tax schedule.

If N is a finite set or if the allocation $(c(\cdot), y(\cdot))$ is not continuous, the set $y(N)$ is not an interval. In this case, the specification of $T(y)$ for $y \notin y(N)$ is somewhat arbitrary; this arbitrariness introduces the possibility that, at a boundary point of $y(N)$, the specified tax schedule may not be differentiable from the right. However, it is always possible to specify $T(\cdot)$ so that the right-hand derivative $\tau(y)$ exists for all y and satisfies (4.15) for all $n \in N$.

for all $n \in N$.[12] If Property D holds, (4.5) implies that $T(\cdot)$ is strictly increasing on $y(N)$. Incentive compatibility requires that, for all $n \in N$, $y(n)$ maximizes $u(y - T(y), y, n)$. If $T(\cdot)$ is differentiable, the first-order conditions for this maximization require that

$$u_c(c(n), y(n), n)(1 - \tau(y(n))) + u_y(c(n), y(n), n) \leq 0, \tag{4.14}$$

with equality unless $y(n) = 0$ or $c(n) = 0$. If Property D holds, the possibility that $c(n) = 0$ can be ruled out because (4.7) implies $c(n^0) > 0$ and, by the monotonicity of $y(\cdot)$ and the co-monotonicity of $c(\cdot)$ and $y(\cdot)$, one has $c(n) \geq c(n^0)$ for $n > n^0$. Thus, (4.14) holds with equality whenever $y(n) > 0$. By a rearrangement of terms, it follows that

$$\tau(y(n)) = \frac{u_c(c(n), y(n), n) + u_y(c(n), y(n), n)}{u_c(c(n), y(n), n)} \tag{4.15}$$

whenever $y(n) > 0$. For $n \in (\hat{n}, n^1]$, $y(n) > 0$ follows from (4.8). From (4.15) and (2.10), one sees that statement A of Theorem 4.1 is equivalent to the allocation $(c(\cdot), y(\cdot))$ and the associated indirect utility function $v(\cdot)$ exhibiting Property A. Similarly, statements B and C are equivalent to the allocation $(c(\cdot), y(\cdot))$ and the associated indirect utility function $v(\cdot)$ exhibiting Properties B and C. ■

I claim that RC, ND, SSCC, and DR are sufficient for any solution to the optimal tax problem to exhibit Properties A - D, so the optimal tax schedule is characterized by Theorem 4.1. Section 4 will establish this claim for a type distribution with finite support, Section 5 for a type distribution with a continuous density whose support is an interval. In this case, one actually obtains the stronger version of Property B whereby, on any compact set of productivity parameters for which $y(n) > 0$ and $F([n^0, n]) \in (0, 1)$, the pairs $(c(n), y(n))$ are distorted downward and bounded away from efficiency.[13]

Before going into details, I briefly explain the proof strategy. Following the line of argument that was developed by Matthews and Moore (1987) for a monopoly problem, I will study the modified problem which is obtained if the requirement of incentive compatibility is weakened to *downward incentive compatibility* and, in addition, a weak monotonicity requirement is

[12] If there is bunching, with $y(n) = y(n')$ for distinct types n and n', incentive compatibility dictates that $c(n) = c(n')$. In this case, (4.13) yields $T(y(n)) = T(y(n'))$, so there is no problem in thinking about T as a function of y.

[13] In Hellwig (2007), I extend these results to cover "mixed" distributions involving mass points as well as a continuous part.

imposed on consumption. I refer to this modified problem as the *weakly relaxed income tax problem*. The word "relaxed" refers to the fact that, under SSCC, incentive compatibility of an allocation implies monotonicity, so downward incentive compatibility and consumption monotonicity together are weaker than incentive compatibility. The word "weakly" refers to the fact that upward incentive compatibility is not just dropped, but is replaced by consumption monotonicity.[14]

In the weakly relaxed income tax problem, the implications of redistribution concerns and incentive constraints are easier to disentangle than in the optimal income tax problem itself. In the end, the analysis also shows that, under the given assumptions, both problems have the same solutions.

The argument proceeds in several distinct steps: First, if consumption is constant on some set of types, then the output requirement must also be constant on this set. Otherwise, by downward incentive compatibility, the higher types in this set would have lower output requirements. Part (a) of condition DR implies that this cannot be optimal because one could raise welfare by equalizing these output requirements.

Second, in the absence of upward incentive constraints, the allocation cannot be distorted upward from efficiency. If, for type n, the pair $(c(n), y(n))$ were distorted upward from efficiency, one could reduce both $c(n)$ and $y(n)$ in such a way that the difference $c(n) - y(n)$ becomes smaller while the utility $v(n) = u(c(n), y(n), n)$ is unchanged. By SSCC, such a reduction would not affect downward incentive compatibility. The decrease in $c(n) - y(n)$ could be used to raise welfare by making, e.g., types near the top of the type distribution, better off.

Third, at any n below the top, if $c(n) > 0$ and $F([n^0, n]) > 0$, then either consumption monotonicity or adjacent downward incentive compatibility must be strictly binding in the sense that the corresponding Kuhn-Tucker multipliers or Pontryagin costate variables are nonzero. Otherwise, the pair $(c(n), y(n))$ would be efficient for type n. Because, for $n' > n$, the outcome $(c(n'), y(n'))$ is not distorted upward from efficiency for these types, the premises of part (b) of condition DR would be satisfied. Therefore, some

[14] In contrast, Matthews and Moore (1987) study a *relaxed problem* involving only downward incentive constraints. In their general discussion of how to simplify global incentive constraints, they mention the possibility of using monotonicity in combination with adjacent downward incentive compatibility; see in particular their fn. 15, p. 447. Their discussion is taken up again by Besley and Coate (1995, p. 197). In the context of optimal income taxation, Hellwig (2005/2007) shows that the relaxed problem involving only downward incentive constraints is equivalent to the optimal income tax problem if the utility function exhibits an additional property of weakly decreasing consumption-specific risk aversion.

additional redistribution from adjacent higher types to type n would raise welfare.

Fourth, at any n between n^0 and n^1, adjacent downward incentive compatibility is satisfied with equality. For adjacent types that are assigned the same outcome pairs, this assertion is trivial. For adjacent types that are assigned different outcome pairs, consumption monotonicity is not binding, so, by the preceding argument, adjacent downward incentive compatibility must be.

Because adjacent downward incentive compatibility is satisfied with equality for all n, any increase in $c(\cdot)$ must be accompanied by an increase in $y(\cdot)$ and *vice versa*. The functions $c(\cdot)$ and $y(\cdot)$ are thus co-monotonic.

Given the monotonicity of $c(\cdot)$ and $y(\cdot)$, SSCC implies that, if an agent is indifferent between the outcome assigned to him and the outcome assigned to an adjacent lower type, then an agent with the adjacent lower type is happy to receive the outcome that is assigned to him, rather than receiving the outcome that is assigned to the adjacent higher type. The allocation thus satisfies adjacent upward as well as downward incentive compatibility. By monotonicity and SSCC, this implies that the allocation is actually incentive-compatible.

The remaining arguments are straightforward:

Because downward incentive compatibility is everywhere weakly binding, the indifference curve of any type $n \in N$ in the point $(c(n), y(n))$ must be tangent to the image set of the function $(c(\cdot), y(\cdot))$ to the left of this point. In the absence of upward distortions from efficiency, the slope $\frac{dc}{dy}$ of this indifference curve to the left of $(c(n), y(n))$ is less than one. Therefore, any increase in $y(\cdot)$ is accompanied by an increase in $y(\cdot) - c(\cdot)$ and *vice versa*, i.e., the functions $y(\cdot)$, and $y(\cdot) - c(\cdot)$ are co-monotonic.

Given this co-monotonicity property, the image set of the function $(c(\cdot), y(\cdot))$ crosses the 45-degree line at most once. Thus, $c(n^0) = 0$ would imply $c(n) - y(n) \leq 0$ for all n, with a strict inequality if $y(n) > 0$. By ND, such an allocation would be dominated by the laissez-faire allocation. Therefore, one must have $c(n^0) > 0$ and, by monotonicity, $c(n) > 0$ for all $n \in N$.

Because consumption is strictly positive, the third step of the argument implies that, for $n < n^1$, one of the Kuhn-Tucker multipliers or Pontryagin costate variables corresponding to downward incentive compatibility and consumption monotonicity must be nonzero if $F([n^0, n]) > 0$. If $y(n) > 0$, this implies that the outcome pair $(c(n), y(n))$ is distorted downward from efficiency.

All these considerations refer to the weakly relaxed income tax problem. However, being incentive-compatible, any solution to the weakly relaxed income tax problem thus lies in the constraint set of the optimal income tax problem. Because, under SSCC, any incentive-compatible allocation is in fact downward incentive-compatible and nondecreasing, the constraint set of the optimal income tax problem is actually a subset of the constraint set of the weakly relaxed income tax problem. Because the objective functions are the same, it follows that any solution to the weakly relaxed income tax problem which lies in the constraint set of the optimal income tax problem must also be a solution to the optimal income tax problem. Because all solutions to the optimal income tax problem generate the same welfare, any other solution to this problem will also be a solution to the weakly relaxed income tax problem. The optimal income tax problem and the weakly relaxed income tax problem thus have the same solutions.

5 The Case of Finitely Many Types

In this section, I consider the optimal income tax problem when the type set is finite. Without loss of generality, I set $N = \{n_1, n_2, ..., n_m\}$, where $n^0 = n_1 < ... < n_m = n^1$. I also write $f_i := F(\{n_i\}) > 0$. An allocation $(c(\cdot), y(\cdot))$ is identified with a sequence $\{(c_i, y_i)\}_{i=1}^m$ such that $(c_i, y_i) = (c(n_i), y(n_i))$ for $i = 1, ..., m$. The optimal income tax problem is to choose $\{(c_i, y_i)\}_{i=1}^m$ so as to maximize

$$\max_{\{(c_i,y_i)\}_{i=1}^m} \sum_{i=1}^m u(c_i, y_i, n_i)\ f_i \tag{5.1}$$

subject to the feasibility condition

$$\sum_{i=1}^m (y_i - c_i)\ f_i \geq 0, \tag{5.2}$$

and the incentive compatibility the condition that

$$u(c_i, y_i, n_i) \geq u(c_k, y_k, n_i) \tag{5.3}$$

for all i and k. In the remainder of this section, I will prove:

Theorem 5.1 *If the type set is finite, any solution to the optimal income tax problem exibits Properties A - D.*

To prove this theorem, I study the *weakly relaxed income tax problem, i.e.,* the problem of choosing $\{(c_i, y_i)\}_{i=1}^m$ so as to maximize (5.1) subject to the feasibility condition (5.2), the *downward incentive compatibility* condition that (5.3) hold for all i and $k < i$, and the *consumption monotonicity* condition that

$$c_i \geq c_{i-1} \tag{5.4}$$

for all i. The proof strategy is to show (i) that any solution to the weakly relaxed income tax problem exhibits Properties A - D, and (ii) that the solution sets of the optimal and the weakly relaxed income tax problems are the same.

Under SSCC, the downward incentive compatibility constraint in the formulation of the weakly relaxed income tax problem can actually be weakened to an *adjacent* downward incentive compatibility constraint.

Lemma 5.2 *An allocation $\{(c_i, y_i)\}_{i=1}^m$ with nondecreasing consumption is downward incentive-compatible if and only if it satisfies the adjacent downward incentive constraint*

$$u(c_i, y_i, n_i) \geq u(c_{i-1}, y_{i-1}, n_i) \tag{5.5}$$

for all i.

Proof. The "only if" statement is trivial. To prove the "if" statement, let $\{(c_i, y_i)\}_{i=1}^m$ satisfy adjacent downward incentive compatibility and consumption monotonicity. To show that $\{(c_i, y_i)\}_{i=1}^m$ satisfies downward incentive compatibility, I will fix i and use an induction on $k < i$. For $i = 1$, there is nothing to prove. Suppose, therefore, that $i > 1$. Adjacent downward incentive compatibility implies that (5.3) holds for $k = i - 1$. For the induction, suppose that (5.3) holds for some $k < i$ and consider the validity of (5.3) for $k + 1$. If $y_{i-(k+1)} \geq y_{i-k}$, then, by consumption monotonicity and RC, one has $u(c_{i-k}, y_{i-k}, n_i) \geq u(c_{i-(k+1)}, y_{i-(k+1)}, n_i)$, and the validity of (5.3) for $k + 1$ follows from the induction hypothesis. If $y_{i-(k+1)} < y_{i-k}$, then, by consumption monotonicity and adjacent downward incntive compatibility, one has $(c_{i-(k+1)}, y_{i-(k+1)}) \leq (c_{i-k}, y_{i-k})$ and $u(c_{i-k}, y_{i-k}, n_{i-k}) \geq u(c_{i-(k+1)}, y_{i-(k+1)}, n_{i-k})$. By SSCC, it follows that

$$u(c_{i-k}, y_{i-k}, n_i) \geq u(c_{i-(k+1)}, y_{i-(k+1)}, n_i).$$

The validity of (5.3) for $k + 1$ again follows from the induction hypothesis. The induction is thus complete. ■

The weakly relaxed income tax problem is thus reduced to the problem of maximizing (5.1) subject to (5.2), (5.5), and (5.4). The Lagrangian for this problem can be written as

$$\sum_{i=1}^{m} u(c_i, y_i, n_i)\ f_i + \lambda \sum_{i=1}^{m} (y_i - c_i)\ f_i$$
$$+ \sum_{i=2}^{m} \mu_i [u(c_i, y_i, n_i) - u(c_{i-1}, y_{i-1}, n_i)] + \sum_{i=2}^{m} \nu_i (c_i - c_{i-1}), \quad (5.6)$$

where λ and μ_i, ν_i, $i = 2, ...m$, are nonnegative multipliers for the constraints (5.2), (5.5), and (5.4). The Kuhn-Tucker conditions for a solution are:

$$u_c(c_i, y_i, n_i)(f_i + \mu_i) - \lambda f_i - \mu_{i+1} u_c(c_i, y_i, n_{i+1}) + \nu_i - \nu_{i+1} \leq 0 \quad (5.7)$$

for c_i, $i = 1, ..., m$, with a strict inequality only if $c_i = 0$, and

$$u_y(c_i, y_i, n_i)(f_i + \mu_i) + \lambda f_i - \mu_{i+1} u_y(c_i, y_i, n_{i+1}) \leq 0 \quad (5.8)$$

for y_i, $i = 1, ..., m$, with a strict inequality only if $y_i = 0$.[15] Moreover,

$$\lambda \sum_{i=1}^{m} (y_i - c_i) f_i = 0, \quad (5.9)$$

$$\mu_i [u(c_i, y_i, n_i) - u(c_{i-1}, y_{i-1}, n_i)] = 0, \quad (5.10)$$

and

$$\nu_i (c_i - c_{i-1}) = 0 \quad (5.11)$$

for $i = 2, ..., m$. In the remainder of this section, the allocation $\{(c_i, y_i)\}_{i=1}^m$ is taken to be a solution to problem of maximizing (5.1) subject to (5.2), (5.5), (5.4); λ, μ_i, and ν_i, $i = 2, ...m$, are the associated Kuhn-Tucker multipliers in (5.6) - (5.11).

I first show that people with the same consumption must also provide the same output.

Lemma 5.3 *For any k, $c_k = c_{k-1}$ implies $y_k = y_{k-1}$.*

Proof. Suppose that the lemma is false, and let k be such that $c_k = c_{k-1}$ and $y_k \neq y_{k-1}$. Then downward incentive compatibility implies $y_k < y_{k-1}$, hence

$$u(c_k, y_k, n_k) > u(c_{k-1}, y_{k-1}, n_k). \quad (5.12)$$

[15] If $i = m$, (5.7) and (5.8) hold with $\mu_{i+1} = \nu_{i+1} = 0$.

By (5.10), it follows that $\mu_k = 0$. Therefore (5.8) becomes:

$$u_y(c_{k-1}, y_{k-1}, n_{k-1})(f_{k-1} + \mu_{k-1}) + \lambda f_{k-1} = 0 \tag{5.13}$$

for $i = k - 1$ and

$$u_y(c_k, y_k, n_k) f_k + \lambda f_k - \mu_{k+1} u_y(c_k, y_k, n_{k+1}) \leq 0 \tag{5.14}$$

for $i = k$. From (5.13) and (5.14), one obtains

$$u_y(c_{k-1}, y_{k-1}, n_{k-1}) \geq -\lambda \geq u_y(c_k, y_k, n_k), \tag{5.15}$$

contrary to part (a) of DR. The assumption that $c_k = c_{k+1}$ and $y_k \neq y_{k+1}$ for some k has thus led to a contradiction and must be false. ■

The next lemma shows that, in a solution to problem (5.1), consumption and output provision are never distorted upward from efficiency. To simplify the notation, I write (c_k^*, y_k^*) for the pair $(c^*(n_k, v(n_k)), y^*(n_k, v(n_k)))$, which provides type n_k with the utility $v(n_k) = u(c_k, y_k, n_k)$ at the lowest net resource requirement.

Lemma 5.4 *For any k, $(c_k, y_k) \leq (c_k^*, y_k^*)$.*

Proof. If the lemma is false, one has $c_{\hat{k}} > c_{\hat{k}}^*$ or $y_{\hat{k}} > y_{\hat{k}}^*$ for some $\hat{k}$. By RC and the fact that

$$u(c_{\hat{k}}, y_{\hat{k}}, n_{\hat{k}}) = v(n_{\hat{k}}) = u(c_{\hat{k}}^*, y_{\hat{k}}^*, n_{\hat{k}}),$$

one must actually have $(c_k, y_k) \gg (c_k^*, y_k^*)$. Being upward distorted from efficiency, they satisfy the inequality

$$u_c(c_{\hat{k}}, y_{\hat{k}}, n_{\hat{k}}) + u_y(c_{\hat{k}}, y_{\hat{k}}, n_{\hat{k}}) < 0. \tag{5.16}$$

Let $I(\hat{k})$ be the set of indices i with the same consumption as index $\hat{k}$, and let $\hat{\imath} := \min I(\hat{k})$. By Lemma 5.3, $(c_{\hat{\imath}}, y_{\hat{\imath}}) = (c_{\hat{k}}, y_{\hat{k}})$. Moreover, by SSCC, (5.16) implies

$$u_c(c_{\hat{\imath}}, y_{\hat{\imath}}, n_{\hat{k}}) + u_y(c_{\hat{\imath}}, y_{\hat{\imath}}, n_{\hat{k}}) < 0. \tag{5.17}$$

Because $(c_{\hat{\imath}}, y_{\hat{\imath}}) = (c_{\hat{k}}, y_{\hat{k}}) \gg (c_k^*, y_k^*)$, $c_{\hat{\imath}}$ and $y_{\hat{\imath}}$ are both strictly positive. For $k = \hat{\imath}$, the first-order conditions (5.7) and (5.8) can therefore be written as

$$u_c(c_{\hat{\imath}}, y_{\hat{\imath}}, n_{\hat{\imath}})(f_{\hat{\imath}} + \mu_{\hat{\imath}}) - \lambda f_{\hat{\imath}} - \mu_{\hat{\imath}+1} u_c(c_{\hat{\imath}}, y_{\hat{\imath}}, n_{\hat{\imath}+1}) = \nu_{\hat{\imath}+1} - \nu_{\hat{\imath}} \tag{5.18}$$

and

$$u_y(c_{\hat{\imath}}, y_{\hat{\imath}}, n_{\hat{\imath}})(f_{\hat{\imath}} + \mu_{\hat{\imath}}) + \lambda f_{\hat{\imath}} - \mu_{\hat{\imath}+1} u_y(c_{\hat{\imath}}, y_{\hat{\imath}}, n_{\hat{\imath}+1}) = 0. \tag{5.19}$$

If one solves (5.19) for the multiplier $\mu_{\hat{\imath}}$, substitutes into (5.18), and rearranges terms, one obtains:

$$\begin{aligned} \nu_{\hat{\imath}+1} - \nu_{\hat{\imath}} \quad = \quad & -\lambda \frac{u_c(c_{\hat{\imath}}, y_{\hat{\imath}}, n_{\hat{\imath}}) + u_y(c_{\hat{\imath}}, y_{\hat{\imath}}, n_{\hat{\imath}})}{u_y(c_{\hat{\imath}}, y_{\hat{\imath}}, n_{\hat{\imath}})} f_i \\ & + \mu_{\hat{\imath}+1} \left(\frac{u_c(c_{\hat{\imath}}, y_{\hat{\imath}}, n_{\hat{\imath}})}{u_y(c_{\hat{\imath}}, y_{\hat{\imath}}, n_{\hat{\imath}})} u_y(c_{\hat{\imath}}, y_{\hat{\imath}}, n_{\hat{\imath}+1}) - u_c(c_{\hat{\imath}}, y_{\hat{\imath}}, n_{\hat{\imath}+1}) \right). \end{aligned} \tag{5.20}$$

By (5.17) and RC, the first term on the right-hand side of (5.20) is negative. By SSCC and the nonnegativity of $\mu_{\hat{\imath}+1}$, the second term is nonpositive. Thus, (5.20) implies $\nu_{\hat{\imath}+1} - \nu_{\hat{\imath}} < 0$. However, $\nu_{\hat{\imath}+1}$ is nonnegative, and, by the definition of $\hat{\imath}$ and (5.11), $\nu_{\hat{\imath}}$ is equal to zero. The assumption that $c_k > c_k^*$ or $y_k > y_k^*$ for some k has thus led to a contradiction and must be false. ■

Efficiency is obtained if neither the downward incentive compatibility constraint nor the consumption monotonicity constraint for the adjacent higher type is binding. This is the point of:

Lemma 5.5 *For any k, $\mu_{k+1} = \nu_{k+1} = 0$ implies $(c_k, y_k) = (c_k^*, y_k^*)$. In this case, moreover, $(c_k, y_k) \gg (0, 0)$ implies $c_i < c_k$ for $i < k$.*

Proof. If $\mu_{k+1} = \nu_{k+1} = 0$, the first-order conditions (5.7) and (5.8) become

$$u_c(c_k, y_k, n_k)(f_k + \mu_k) - \lambda f_k + \nu_k \leq 0, \tag{5.21}$$

with equality unless $c_k = 0$, and

$$u_y(c_k, y_k, n_k)(f_k + \mu_k) + \lambda f_k \leq 0, \tag{5.22}$$

with equality unless $y_{\hat{k}} = 0$. Upon adding these inequalities and rearranging terms, one obtains

$$u_c(c_k, y_k, n_k) + u_y(c_k, y_k, n_k) \leq -\frac{\nu_k}{f_k + \mu_k}, \tag{5.23}$$

with equality unless $c_k = 0$ or $y_k = 0$. Because the right-hand side of (5.23) is nonpositive, it follows that $(c_k, y_k) \geq (c_k^*, y_k^*)$. By Lemma 5.4, therefore, $(c_k, y_k) = (c_k^*, y_k^*)$.

If $(c_k, y_k) = (c_k^*, y_k^*) \gg (0, 0)$, the first-order condition for efficiency yields

$$u_c(c_k, y_k, n_k) + u_y(c_k, y_k, n_k) = 0.$$

For $i < k$, therefore, SSCC implies

$$u_c(c_k, y_k, n_i) + u_y(c_k, y_k, n_i) < 0.$$

By Lemma 5.4, it follows that $(c_i, y_i) \neq (c_k y_k)$. By Lemma 5.3 and consumption monotonicity, it follows that $c_i < c_k$. ■

For $k = m$, the premise of Lemma 5.5 is automatically fulfilled. This observation yields:

Corollary 5.6 $(c_m, y_m) = (c_m^*, y_m^*)$. *Moreover,* $(c_m, y_m) \gg (0, 0)$ *implies* $c_{m-1} < c_m$.

The following lemma shows that, below the top, the premise of Lemma 5.5 is *never* satisfied. Part (b) of condition DR is crucial for this result.

Lemma 5.7 *For any* $k < m$, *if* $c_{k+1} > 0$, *then at least one of the multipliers* μ_{k+1}, ν_{k+1} *is nonzero.*

Proof. If the lemma is false, then, for some $\hat{k} < m$, one has $c_{\hat{k}+1} > 0$ and $\mu_{\hat{k}+1} = \nu_{\hat{k}+1} = 0$. By Lemma 5.5, one has $(c_{\hat{k}}, y_{\hat{k}}) = (c_{\hat{k}}^*, y_{\hat{k}}^*)$. By downward incentive compatibility and consumption monotonicity and by Lemma 5.4, one also has $u(c_{\hat{k}+1}, y_{\hat{k}+1}, n_{\hat{k}+1}) \geq u(c_{\hat{k}}, y_{\hat{k}}, n_{\hat{k}+1})$, $c_{\hat{k}+1} \geq c_{\hat{k}}$, and $(c_{\hat{k}+1}, y_{\hat{k}+1}) \leq (c_{\hat{k}+1}^*, y_{\hat{k}+1}^*)$. Therefore, part (b) of condition DR implies

$$\left|u_y(c_{\hat{k}}, y_{\hat{k}}, n_{\hat{k}})\right| > \min(u_c(c_{\hat{k}+1}, y_{\hat{k}+1}, n_{\hat{k}+1}), \left|u_y(c_{\hat{k}+1}, y_{\hat{k}+1}, n_{\hat{k}+1})\right|) \text{ if } y_{\hat{k}} > 0, \tag{5.24}$$

and

$$u_c(c_{\hat{k}}, y_{\hat{k}}, n_{\hat{k}}) > \min(u_c(c_{\hat{k}+1}, y_{\hat{k}+1}, n_{\hat{k}+1}), \left|u_y(c_{\hat{k}+1}, y_{\hat{k}+1}, n_{\hat{k}+1})\right|) \text{ if } y_{\hat{k}} = 0. \tag{5.25}$$

If $y_{\hat{k}} > 0$, (5.22) holds as an equation, and one has

$$\left|u_y(c_{\hat{k}}, y_{\hat{k}}, n_{\hat{k}})\right| = \frac{f_{\hat{k}}}{f_{\hat{k}} + \mu_{\hat{k}}} \lambda \leq \lambda, \tag{5.26}$$

then (5.24) implies

$$\lambda > \min(u_c(c_{\hat{k}+1}, y_{\hat{k}+1}, n_{\hat{k}+1}), \left|u_y(c_{\hat{k}+1}, y_{\hat{k}+1}, n_{\hat{k}+1})\right|). \tag{5.27}$$

If $y_{\hat{k}} = 0$, (5.27) is still obtained, this time from (5.25) and the observation that (5.21) yields

$$u_c(c_{\hat{k}}, y_{\hat{k}}, n_{\hat{k}}) \leq \frac{f_{\hat{k}}}{f_{\hat{k}} + \mu_{\hat{k}}}\lambda - \nu_{\hat{k}} \leq \lambda. \tag{5.28}$$

(5.27) thus holds regardless of whether $y_{\hat{k}}$ is positive or zero.

However, because $c_{\hat{k}+1} > 0$, $\mu_{\hat{k}+1} = \nu_{\hat{k}+1} = 0$, $\mu_{\hat{k}+2} u_y(c_{\hat{k}+1}, y_{\hat{k}+1}, n_{\hat{k}+2}) \leq 0$, and $\nu_{\hat{k}+2} \leq 0$, the first-order conditions (5.7) and (5.8) for $i = \hat{k}+1$ yield $u_c(c_{\hat{k}+1}, y_{\hat{k}+1}, n_{\hat{k}+1}) \geq \lambda$ and $u_y(c_{\hat{k}+1}, y_{\hat{k}+1}, n_{\hat{k}+1}) \leq -\lambda$, hence

$$\min(u_c(c_{\hat{k}+1}, y_{\hat{k}+1}, n_{\hat{k}+1}), \left|u_y(c_{\hat{k}+1}, y_{\hat{k}+1}, n_{\hat{k}+1})\right|) \geq \lambda, \tag{5.29}$$

which is incompatible with (5.27). The assumption that $c_{\hat{k}+1} > 0$ and $\mu_{\hat{k}+1} = \nu_{\hat{k}+1} = 0$ has thus led to a contradiction and must be false. ■

On the basis of Lemma 5.7, one easily finds that adjacent downward incentive constraints hold everywhere with equality.

Lemma 5.8 *For any $k < m$,*

$$v(n_{k+1}) = u(c_k, y_k, n_{k+1}). \tag{5.30}$$

Proof. If the lemma is false, there exists $\hat{k} < m$, such that

$$v(n_{\hat{k}+1}) = u(c_{\hat{k}+1}, y_{\hat{k}+1}, n_{\hat{k}+1}) > u(c_{\hat{k}}, y_{\hat{k}}, n_{\hat{k}+1}). \tag{5.31}$$

(The reverse inequality is ruled out by downward incentive compatibility.) Thus, $(c_{\hat{k}+1}, y_{\hat{k}+1}) \neq (c_{\hat{k}}, y_{\hat{k}})$. By Lemma 5.3 and consumption monotonicity, it follows that $c_{\hat{k}+1} > c_{\hat{k}}$. By (5.10) and (5.11), therefore, $\mu_{\hat{k}+1} = 0$ and $\nu_{\hat{k}+1} = 0$. By Lemma 5.7, this is only possible if $c_{\hat{k}+1} = 0$. But then, one cannot have $c_{\hat{k}+1} > c_{\hat{k}}$. ■

Equation (5.30) implies that, for type n_{k+1}, the point (c_k, y_k) lies on the indifference curve through (c_{k+1}, y_{k+1}). By RC and consumption monotonicity, this implies:

Corollary 5.9 *The sequences $\{c_i\}_{i=1}^m$ and $\{y_i\}_{i=1}^m$ are nondecreasing and co-monotonic.*

By consumption monotonicity, and SSCC, (5.30) also implies

$$v(n_k) \geq u(c_{k+1}, y_{k+1}, n_k), \tag{5.32}$$

i.e., the allocation $\{(c_i, y_i)\}_{i=1}^m$ satisfies adjacent upward incentive compatibility, as well as downward incentive compatibility. By standard arguments, based on monotonicity and SSCC, this yields:

Corollary 5.10 *The allocation $\{(c_i, y_i)\}_{i=1}^m$ is incentive-compatible.*

By the argument of Guesnerie and Seade (1982), Lemmas 5.8 and 5.4 combined yield the desired co-monotonicity properties.

Lemma 5.11 *The sequence $\{y_i - c_i\}_{i=1}^m$ is nondecreasing and co-monotonic with $\{c_i\}_{i=1}^m$ and $\{y_i\}_{i=1}^m$.*

Proof. By Lemma 5.4 in combination with the monotonicity and strict quasiconcavity of u, the slope of the indifference curve of type n_{k+1} through (c_{k+1}, y_{k+1}) is less than one at any point (c, y) below (c_{k+1}, y_{k+1}). The lemma follows from (5.30). ■

Lemma 5.12 *For all k, $c_k > 0$.*

Proof. I will show that, if the lemma is false, the allocation $\{(c_i, y_i)\}_{i=1}^m$ is dominated by the laissez-faire allocation $\{(y_i^{LF}, y_i^{LF})\}_{i=1}^m$, where, for any i, $y_i^{LF} := \arg\max u(y, y, n_i)$. This is incompatible with the assumption that $\{(c_i, y_i)\}_{i=1}^m$ is a solution to the problem of the weakly relaxed income tax problem because, trivially, under SSCC, the laissez-faire allocation satisfies feasibility, downward incentive compatibility, and consumption monotonicity.

If the lemma is false, then, by consumption monotonicity, there exists $\hat{k} \in \{1, ..., m\}$ such that $c_k = 0$ for $k = 1, ..., \hat{k}$. For any $k \leq \hat{k}$, RC implies

$$u(c_k, y_k, n_k) \leq u(0, 0, n_k), \tag{5.33}$$

hence, also

$$u(c_k, y_k, n_k) \leq u(y_k^{LF}, y_k^{LF}, n_k). \tag{5.34}$$

If $\hat{k} = m$, condition ND implies that, for $k = m$, the inequality in (5.34) is strict. Summation of (5.34) over $k = 1, ..., m$ then yields

$$\sum_{k=1}^m u(c_k, y_k, n_k) f_k < \sum_{k=1}^m u(y_k^{LF}, y_k^{LF}, n_k) f_k. \tag{5.35}$$

If $\hat{k} < m$, then, for $k > \hat{k}$, Lemma 5.11 implies $c_k - y_k \geq c_{\hat{k}+1} - y_{\hat{k}+1}$ and, since $c_{\hat{k}+1} > c_{\hat{k}}$, $c_{\hat{k}+1} - y_{\hat{k}+1} < -y_{\hat{k}} \leq 0$. By RC, it follows that

$$u(c_k, y_k, n_k) < u(y_k^{LF}, y_k^{LF}, n_k) \tag{5.36}$$

for $k = \hat{k}+1, ..., m$. Summation of (5.34) over $k \leq \hat{k}$ and of (5.36) over $k > \hat{k}$ again yields (5.35).

In either case, if $\hat{k} = m$ and if $\hat{k} < m$, one obtains a contradiction the assumption that $\{(c_i, y_i)\}_{i=1}^m$ maximizes (5.1) subject to (5.2) - (5.4). The assumption that $c_k = 0$ for some k must therefore be false. ■

To conclude the argument, I finally show that, below the top, consumption and output provision must be distorted downward from efficiency. The argument again relies on Lemma 5.7 and, thereby, on part (b) of condition DR.

Lemma 5.13 *For any $k < m$, $y_k > 0$ implies $(c_k, y_k) \ll (c_k^*, y_k^*)$.*

Proof. Suppose that the lemma is false. Then, for some $\hat{k} < m$, one has $c_{\hat{k}} > 0$, $y_{\hat{k}} > 0$, and $c_{\hat{k}} \geq c_{\hat{k}}^*$ or $y_{\hat{k}} \geq y_{\hat{k}}^*$. By Lemma 5.4, one must actually have $(c_{\hat{k}}, y_{\hat{k}}) = (c_{\hat{k}}^*, y_{\hat{k}}^*)$. Positivity of $(c_{\hat{k}}, y_{\hat{k}})$ implies that, for $i = \hat{k}$, conditions (5.8) and (5.7) must hold as equations. Upon adding these equations, one obtains

$$\begin{aligned} &[u_c(c_{\hat{k}}, y_{\hat{k}}, n_{\hat{k}}) + u_y(c_{\hat{k}}, y_{\hat{k}}, n_{\hat{k}})](f_{\hat{k}} + \mu_{\hat{k}}) + \nu_{\hat{k}} \\ &= \mu_{\hat{k}+1}[u_c(c_{\hat{k}}, y_{\hat{k}}, n_{\hat{k}+1}) + u_y(c_{\hat{k}}, y_{\hat{k}}, n_{\hat{k}+1})] + \nu_{\hat{k}+1}. \end{aligned} \tag{5.37}$$

Because $(c_{\hat{k}}, y_{\hat{k}})$ is strictly positive and efficient, one also obtains

$$u_c(c_{\hat{k}}, y_{\hat{k}}, n_{\hat{k}}) + u_y(c_{\hat{k}}, y_{\hat{k}}, n_{\hat{k}}) = 0. \tag{5.38}$$

The first term on the left-hand side of (5.37) is thus equal to zero. The second term is also equal to zero. For suppose that $\nu_{\hat{k}} > 0$. By (5.11) and Lemma 5.3, one then has $c_{\hat{k}-1} = c_{\hat{k}}$ and $y_{\hat{k}-1} = y_{\hat{k}}$. At the same time, (5.38) and SSCC imply

$$u_c(c_{\hat{k}}, y_{\hat{k}}, n_{\hat{k}-1}) + u_y(c_{\hat{k}}, y_{\hat{k}}, n_{\hat{k}-1}) < 0,$$

so that, for type $n_{\hat{k}-1}$, $(c_{\hat{k}}, y_{\hat{k}})$ is distorted upwards from efficiency. By Lemma 5.4, it follows that $(c_{\hat{k}-1}, y_{\hat{k}-1}) \ll (c_{\hat{k}}, y_{\hat{k}})$. The assumption that $\nu_{\hat{k}} > 0$ has thus led to a contradiction and must be false. The left-hand side of (5.37) is thus equal to zero.

By SSCC, (5.38) also implies that $u_c(c_{\hat{k}}, y_{\hat{k}}, n_{\hat{k}+1}) + u_y(c_{\hat{k}}, y_{\hat{k}}, n_{\hat{k}+1}) > 0$. By Lemmas 5.7 and 5.12, it follows that either the first term or the second term on the right-hand side of (5.37) is strictly positive. The other term is nonnegative, because $\mu_{\hat{k}+1}$ and $\nu_{\hat{k}+1}$ are both nonnegative. Therefore, the right-hand side of (5.37) is strictly positive. The assumption that the lemma is false has thus led to a contradiction. ■

The preceding results show that any solution to the weakly relaxed income tax problem exhibits Properties A, B, and D as specified in Section 3: Property A holds by Corollary 5.6, Property B by Lemma 5.13, Property D by Lemmas 5.11 and 5.12 and Corollary 5.6. Property C is moot because $F(\{n^0\}) = f_1 > 0$. Given these observations, Theorem 5.1 follows because, by the argument at the end of Section 4, Corollary 5.10 implies that the optimal income tax problem and the weakly relaxed income tax problem have the same solutions.

6 The Case of a Continuum of Types

6.1 The Control Problem

If the type set N is an interval and the type distribution F has a density f, the optimal income tax problem is to choose an allocation $(c(\cdot), y(\cdot))$ with the associated indirect utility function $v(\cdot)$ in order to maximize the integral

$$\int_{n^0}^{n^1} v(n)f(n)dn = \int_{n^0}^{n^1} u(c(n), y(n), n)f(n)dn \tag{6.1}$$

subject to the feasibility condition

$$\int_{n^0}^{n^1} (y(n) - c(n))f(n)dn \leq 0 \tag{6.2}$$

and incentive compatibility.

For technical reasons, I impose the additional condition that the allocation $(c(\cdot), y(\cdot))$ must be piecewise continuously differentiable. Given that $c(\cdot)$ and $y(\cdot)$ are endogenous, this assumption is problematic, but, as shown in Hellwig (2007), it is not actually necessary for the analysis. Here, I impose it here to avoid encumbering the presentation with control-theoretic details that have little to do with the income tax problem.[16] The main result of this section is then formulated as:

[16] In assuming that $c(\cdot)$ and $y(\cdot)$ are piecewise continuously differentiable, I follow the approach of Guesnerie and Laffont (1984); see also Ebert (1992) and Brunner (1993).

Theorem 6.1 *If the type set is an interval and the type distribution has a density that is continuous and strictly positive on this interval, any solution to the optimal income tax problem that is piecewise continuously differentiable exibits Properties A - D.*

I approach the proof of this theorem by looking at the weakly relaxed income tax problem of maximizing (6.1) subject to feasibility, downward incentive compatibility, and consumption monotonicity, i.e., subject to (6.2) and the requirement that the inequalities $u(c(n), y(n), n) \geq u(c(n'), y(n'), n)$ and $c(n) \geq c(n')$ hold for all n and all $n' < n$. As an analogue to Lemma 5.2, the following lemma shows that, as in the finite case, under SSCC, downward incentive compatibility can be weakened to a *local* downward incentive compatibility condition.

Lemma 6.2 *A piecewise continuously differentiable allocation $(c(\cdot), y(\cdot))$ with nondecreasing $c(\cdot)$ is downward incentive-compatible if and only if the associated indirect utility function $v(\cdot)$ is absolutely continuous and its derivative satisfies*

$$v'(n) \geq u_n(c(n), y(n), n) \tag{6.3}$$

for all n.

The proof of lemma 6.2 is given in Appendix B. It is virtually identical to Mirrlees's (1976) proof that, under SSCC, incentive compatibility is equvalent to the requirement that the allocation be nondecreasing on N and that the indirect utility function $v(\cdot)$ satisfy the differential equation $v'(n) = u_n(c(n), y(n), n)$.

Given the restriction to piecewise continuously differentiable allocations, the weakly relaxed problem can be formulated as a standard problem of optimal control with state variables $v(\cdot)$ and $c(\cdot)$ and control variables $y(\cdot)$, $s(\cdot)$, and $q(\cdot)$, where, for any n,

$$s(n) := v'(n) - u_n(c(n), y(n), n), \tag{6.4}$$

and

$$q(n) := c'(n).$$

Condition (6.3) and consumption monotonicity are equivalent to the requirements that $s(n) \geq 0$ and $q(n) \geq 0$. With the restriction to piecewise continuously differentiable allocations, the weakly relaxed income tax problem is therefore equivalent to the problem of choosing $v(\cdot), c(\cdot), y(\cdot), s(\cdot)$, and

$q(\cdot)$ so as to maximize (6.1) subject to (2.6), (6.2), (6.4), and nonnegativity of $s(\cdot)$ and $q(\cdot)$. This control problem has the Hamiltonian

$$\begin{aligned} H(n) &= v(n)f(n) + \lambda(y - c(n))f(n) + \kappa(n)(u(c(n), y, n) - v(n)) \\ &\quad + \varphi(n)[s + u_n(c(n), y, n)] + \psi(n)q, \end{aligned} \tag{6.5}$$

where λ and $\kappa(n)$ are the Lagrange multipliers of the constraints (6.2) and (2.6), and $\varphi(\cdot)$ and $\psi(\cdot)$ are the costate variables associated with the state variables $v(\cdot)$ and $c(\cdot)$.

The requirement that $y(n)$, $s(n)$, and $q(n)$ maximize the Hamiltonian with respect to the controls $y \geq 0, s \geq 0$, and $q \geq 0$ yields the first-order conditions:

$$\lambda f(n) + \kappa(n)u_y(c(n), y(n), n) + \varphi(n)u_{ny}(c(n), y(n), n) \leq 0, \tag{6.6}$$

with equality if $y(n) > 0$;

$$\psi(n) \leq 0, \tag{6.7}$$

with equality if $q(n) > 0$; finally,

$$\varphi(n) \leq 0, \tag{6.8}$$

with equality if $s(n) > 0$. The costate variables $\varphi(\cdot)$ and $\psi(\cdot)$ are absolutely continuous; their derivatives satisfy

$$\varphi'(n) = -f(n) + \kappa(n), \tag{6.9}$$

and

$$\psi'(n) \leq \lambda f(n) - \kappa(n)u_c(c(n), y(n), n) - \varphi(n)u_{nc}(c(n), y(n), n), \tag{6.10}$$

where (6.10) is an equation if $c(n) > 0$. In addition, $\varphi(\cdot)$ and $\psi(\cdot)$ must satisfy the transversality conditions

$$\varphi(n^0) = \varphi(n^1) = 0, \tag{6.11}$$

and

$$\psi(n^0)c(n^0) = \psi(n^1) = 0. \tag{6.12}$$

Finally, one has $\lambda \geq 0$, with

$$\lambda \int_{n^0}^{n^1} (y(n) - c(n))f(n)dn = 0. \tag{6.13}$$

If one uses (6.9) to eliminate $\kappa(n)$ from (6.10) and (6.6), one obtains:

$$\psi'(n) - \lambda f(n) + u_c f(n) + \varphi'(n) u_c + \varphi(n) u_{nc} \leq 0, \tag{6.14}$$

with equality if $c(n) > 0$, and

$$\lambda f(n) + u_y f(n) + \varphi'(n) u_y + \varphi(n) u_{ny} \leq 0 \tag{6.15}$$

with equality if $y(n) > 0$. Except for the fact that $\psi'(n)$ appears in the condition referring to $c(n)$ rather than $y(n)$, these conditions are familiar from Ebert (1992) and Brunner (1993). If $\psi'(n) = 0$, they reduce to the corresponding conditions in Mirrlees (1971, 1976) and Seade (1977, 1982).

Upon combining (6.14) and (6.15) so as to eliminate $\varphi'(n)$, one further obtains

$$\psi'(n) \leq \lambda \frac{u_c + u_y}{u_y} f(n) - \varphi(n)(u_{nc} - \frac{u_c}{u_y} u_{ny}), \tag{6.16}$$

with equality if $c(n) > 0$ and $y(n) > 0$. This is the central condition of the model with a continuous type distribution. If $c(n) > 0$ and $y(n) > 0$ and $\psi'(n) = 0$, it yields the equation

$$\lambda \frac{u_c + u_y}{u_y} f(n) = \varphi(n)(u_{nc} - \frac{u_c}{u_y} u_{ny}), \tag{6.17}$$

which lies at the core of the analysis of Mirrlees and Seade. Because $u_y < 0$, the left-hand side is negative or positive, depending on whether $u_c + u_y$ is positive or negative, i.e. on whether $(c(n), y(n))$ is distorted downward or upward from efficiency. SSCC implies that $u_{nc} - \frac{u_c}{u_y} u_{ny}$ is positive, so the sign of the right-hand side is the same as the sign of $\varphi(n)$.

At this point, Seade (1982) investigates the global properties of φ when treated as a solution to (6.14) and (6.15) in order to show that one cannot have $\varphi(n) > 0$. By contrast, the indirect approach developed here has $\varphi(n) \leq 0$ already from the necessary condition (6.8) for the choice of the slack variable $w(n)$. From (6.17), one immediately knows that $u_c + u_y$ is nonnegative, i.e., that $(c(n), y(n))$ cannot be distorted upward from efficiency. The problem is to show that, between n^0 and n^1, one actually has $\varphi(n) > 0$ and $u_c + u_y > 0$, i.e., that $(c(n), y(n))$ is actually distorted downward from efficiency. One must also deal with the possibility of corner solutions for $c(n)$ or $y(n)$ and with the possibility of bunching, implying that $\psi'(n)$ might not be zero.

6.2 Proof of Theorem 6.1

The proof of Theorem 6.1 follows the same line of argument as the proof of Theorem 5.1 in the finite case. I first use part (a) of condition DR to show that people with the same consumption must also provide the same output. This corresponds to Lemma 5.3 in the finite-type case.

Lemma 6.3 *For any n and $n' > n$, in N, $c(n) = c(n')$ implies that $y(n) = y(n')$.*

Proof. Suppose that the lemma is false, and let n, $n' > n$ be such that $c(n) = c(n')$ and $y(n) \neq y(n')$. By consumption monotonicity, $c(\cdot)$ is constant on the interval $[n, n']$. By downward incentive compatibility, therefore, $y(\cdot)$ is nonincreasing on $[n, n']$ and $y(n) > y(n')$. Because $y(\cdot)$ is continuous, there exists a nondegenerate subinterval $[\bar{n}, \bar{n}'] \subset [n, n']$ on which $y(\cdot)$ is strictly decreasing. In the interior of this subinterval, there is slack in adjacent downward incentive constraints, i.e., one must have $s(n'') > 0$. By the first-order condition for $s(n'')$, it follows that $\varphi(n'') = 0$ for $n'' \in (\bar{n}, \bar{n}')$. This in turn implies that $\varphi'(n'') = 0$ for $n'' \in (\bar{n}, \bar{n}')$. Because $y(\cdot)$ is strictly decreasing, one must also have $y(n'') > 0$ for $n'' \in (\bar{n}, \bar{n}')$. For $n'' \in (\bar{n}, \bar{n}')$, condition (6.15) thus takes the form $\lambda f(n'') + u_y f(n'') = 0$. It follows that u_y is constant on $(\bar{n}, \bar{n}')$, contrary to part (a) of condition DR. The assumption that the lemma is false has thus led to a contradiction. ■

The next lemma shows that consumption and output provision are never inefficiently high. This corresponds to Lemma 5.4 in the finite case. The argument presumes that the Lagrange multiplier λ of the feasibility constraint is strictly positive. In the finite case, this is trivially implied by the first-order condition for c_m. In the continuous-type case, the argument is more complicated. A formal statement and proof are given in Appendix C.

Lemma 6.4 *For all $n \in N$, $(c(n), y(n)) \leq (c^*(n), y^*(n))$.*

Proof. If the lemma is false, one has $(c(\hat{n}), y(\hat{n})) \gg (c^*(\hat{n}), y^*(\hat{n}))$ for some $\hat{n} \in N$. Let $\bar{n} \leq \hat{n}$ be the smallest type that has the same consumption as $\hat{n}$. By Lemma 6.3, this type also has the same output requirement as $\hat{n}$, and one has $(c(\bar{n}), y(\bar{n})) = (c(\hat{n}), y(\hat{n}))$. By the same argument as in the proof of Lemma 5.4, one also has

$$u_c(c(\hat{n}), y(\hat{n}), \bar{n}) + u_y(c(\hat{n}), y(\hat{n}), \bar{n}) < 0, \tag{6.18}$$

so, with $(c(\hat{n}), y(\hat{n})) \gg (0, 0)$, one also has $(c(\hat{n}), y(\hat{n})) \gg (c^*(\bar{n}), y^*(\bar{n}))$, hence $(c(\bar{n}), y(\bar{n})) \gg (c^*(\bar{n}), y^*(\bar{n}))$.

By continuity, it follows that, for some $\delta > 0$, one has $(c(n), y(n)) \gg (c^*(n), y^*(n))$ for all $n \in (\bar{n}, \bar{n} + \delta)$. Trivially, also $(c(n), y(n)) \gg 0$ for $n \in (\bar{n}, \bar{n} + \delta)$. On the interval $(\bar{n}, \bar{n} + \delta)$, (6.16) must therefore hold as an equation, i.e., one has

$$\psi'(n) = \lambda \frac{u_c + u_y}{u_y} f(n) - \varphi(n)(u_{nc} - \frac{u_c}{u_y} u_{ny}). \tag{6.19}$$

By RC and the positivity of λ, $(c(n), y(n)) \gg (c^*(n), y^*(n))$ implies that the first term on the right-hand side of (6.19) is strictly positive. By SSCC and (6.8), the second term on the right-hand side of (6.19) is nonnegative. Therefore, $\psi'(n) > 0$ for $n \in (\bar{n}, \bar{n} + \delta)$. Since (6.7) implies $\psi(\bar{n} + \delta) \leq 0$, it follows that $\psi(\bar{n}) < 0$.

Since $\psi(\bar{n}) < 0$ and $c(\bar{n}) = c(\hat{n}) > 0$, the transversality condition (6.12) implies that one must have $\bar{n} > n^0$. For $n \in (n^0, \bar{n})$, the definition of $\bar{n}$ implies $c(n) < c(\bar{n})$. Therefore, one must have $q(n^k) > 0$ and $\psi(n^k) = 0$ for all elements of some sequence $\{n^k\}$ that converges to $\bar{n}$ from below. By continuity, this implies $\psi(\bar{n}) = 0$. The assumption that $(\hat{c}(\hat{n}), \hat{y}(\hat{n})) \gg (\hat{c}^*(\hat{n}), \hat{y}^*(\hat{n}))$ for some $\hat{n} \in N$ thus leads to a contradiction and must be false. ■

Lemma 6.5 *For any $\hat{n} \in (n^0, n^1]$, $\varphi(\hat{n}) = \psi(\hat{n}) = 0$ implies*

$$(c(\hat{n}), y(\hat{n})) = (c^*(\hat{n}), y^*(\hat{n})). \tag{6.20}$$

Moreover, if $(c(\hat{n}), y(\hat{n})) \gg 0$, then $c(n) < \hat{c}(\hat{n})$ for $n < \hat{n}$.

Proof. $\psi(\hat{n}) = 0$ implies that, for some sequence $\{n^k\}$ which converges to $\hat{n}$ from below, one has $\psi'(n^k) \geq 0$ for all k. For any element of this sequence, (6.16) yields

$$\varphi(n^k)(u^k_{nc} - \frac{u^k_c}{u^k_y} u^k_{ny}) \leq \lambda \frac{u^k_c + u^k_y}{u^k_y} f(n^k). \tag{6.21}$$

By the continuity of φ, $\varphi(\hat{n}) = 0$ implies that the left-hand side of (6.21) goes to zero as n^k converges to $\hat{n}$. Since $u^k_y < 0$ for all k, it follows that

$$\lim_{k \to \infty} (u^k_c + u^k_y) \leq 0.$$

Because $c(\cdot)$ and $y(\cdot)$ are continuous, therefore,

$$u_c(c(\hat{n}), y(\hat{n}), \hat{n}) + u_y(c(\hat{n}), y(\hat{n}), \hat{n}) \leq 0 \tag{6.22}$$

and $(c(\hat{n}), y(\hat{n})) \geq (c^*(\hat{n}), y^*(\hat{n}))$. By Lemma 6.4, (6.20) follows.

If $(c(\hat{n}), y(\hat{n})) \gg 0$, the first-order condition for efficiency implies that

$$u_c(c(\hat{n}), y(\hat{n}), \hat{n}) + u_y(c(\hat{n}), y(\hat{n}), \hat{n}) = 0. \tag{6.23}$$

For $n < \hat{n}$, therefore, SSCC implies

$$u_c(c(\hat{n}), y(\hat{n}), n) + u_y(c(\hat{n}), y(\hat{n}), n) < 0,$$

so Lemma 6.4 yields $(c(n), y(n)) \neq (c(\hat{n}), y(\hat{n}))$. By Lemma 6.3 and consumption monotonicity, it follows that $c(n) < \hat{c}(\hat{n})$. ■

As in the finite case, one immediately concludes that there is no distortion at the top:

Corollary 6.6 $(c(n^1), y(n^1)) = (c^*(n^1), y^*(n^1))$. *Moreover,* $(c(n^1), y(n^1)) \gg 0$ *implies* $c(n) < c(n^1)$ *for all* $n < n^1$.

The next lemma uses part (b) of DR to show that, for $n \in (n^0, n^1)$, the premise of Lemma 6.5 cannot be satisfied. This corresponds to Lemma 5.7 in the finite case.

Lemma 6.7 *For any* $n \in (n^0, n^1)$, *if* $c(n) > 0$, *then at least one of the costate variables* $\varphi(n)$, $\psi(n)$ *is nonzero.*

Proof. A complete proof of this lemma is given in Appendix C. To provide at least the gist of the argument, I give a simpler proof here, which applies if there is no bunching. For any $n \in (n^0, n^1)$, I show that, if $q(n') > 0$ for all n' in some neighbourhood of n, then $\varphi(n) < 0$.

If this claim is false, there exists $\hat{n} \in (n^0, n^1)$ such that $q(n') > 0$ for all n' in some neighbourhood of $\hat{n}$, and, moreover, $\varphi(\hat{n}) = 0$. The positivity of q in a neighbourhood of $\hat{n}$ implies that $\psi'(n) = \psi(n) = 0$ and $c(n) > 0$ for all n in some neighbourhood of $\hat{n}$. In this neighbourhood, condition (6.14) holds as an equation and can be written as

$$\varphi'(n) u_c + \varphi(n) u_{nc} = (\lambda - u_c) f(n). \tag{6.24}$$

Because $c(\cdot), y(\cdot)$ and the density $f(\cdot)$ are continuous, it follows that, in the given neighbourhood of $\hat{n}$, $\varphi'(\cdot)$ is also continuous. By (6.7) $\varphi(\hat{n}) = 0$ means that $\varphi(\cdot)$ is maximal at $\hat{n}$. Therefore, one also has $\varphi'(\hat{n}) = 0$. By (6.14), it follows that

$$u_c(c(\hat{n}), y(\hat{n}), \hat{n}) = \lambda. \tag{6.25}$$

If $y(\hat{n})$ is also positive, then, in a neighbourhood of $\hat{n}$, condition (6.15) also holds as an equation, and one obtains

$$|u_y(c(\hat{n}), y(\hat{n}), \hat{n})| = \lambda. \tag{6.26}$$

By Lemmas 6.5 and 6.4, in combination with downward incentive compatibility and consumption monotonicity, one has $(c(\hat{n}), y(\hat{n})) = (c^*(\hat{n}), y^*(\hat{n}))$ and, for any $n > \hat{n}$, $(c(n), y(n)) \leq (c^*(n), y^*(n))$, $u(c(n), y(n), n) \geq u(c(\hat{n}), y(\hat{n}), \hat{n})$, and $c(n) \geq c(\hat{n})$. By part (b) of condition RC, it follows that

$$|u_y(c(\hat{n}), y(\hat{n}), \hat{n})| > \min(u_c(c(n), y(n), n), |u_y(c(n), y(n), n)|) \quad \text{if} \quad y(\hat{n}) > 0 \tag{6.27}$$

and

$$u_c(c(\hat{n}), y(\hat{n}), \hat{n}) > \min(u_c(c(n), y(n), n), |u_y(c(n), y(n), n)|) \quad \text{if} \quad y(\hat{n}) = 0 \tag{6.28}$$

for all $n > \hat{n}$ that are sufficiently close to $\hat{n}$. By (6.25) and (6.26), therefore, one has

$$\lambda > \min(u_c(c(n), y(n), n), |u_y(c(n), y(n), n)|). \tag{6.29}$$

However, in the neighbourhood of $\hat{n}$ where (6.24) holds, one has

$$\varphi' + \varphi \frac{u_{cn}}{u_c} = \frac{\lambda - u_c}{u_c} f.$$

By SSCC and the nonpositivity of φ, it follows that

$$\varphi' + \varphi \frac{u_{yn}}{u_y} \geq \frac{\lambda - u_c}{u_c} f. \tag{6.30}$$

From (6.15), one also has

$$\varphi' + \varphi \frac{u_{yn}}{u_y} \geq \frac{\lambda - |u_y|}{|u_y|} f. \tag{6.31}$$

Upon combining (6.30) and (6.31), one obtains

$$\varphi' + \varphi \frac{u_{yn}}{u_y} \geq \frac{\lambda - \min(u_c, |u_y|)}{\min(u_c, |u_y|)} f. \tag{6.32}$$

Upon multiplying both sides of this inequality by $\exp(\int_{\hat{n}}^{n} \frac{u_{yn}}{u_y} d\nu)$ and integrating, one finds that

$$\varphi(\bar{n}) \exp(\int_{\hat{n}}^{\bar{n}} \frac{u_{yn}}{u_y} d\nu) \geq \int_{\hat{n}}^{\bar{n}} \frac{\lambda - \min(u_c, |u_y|)}{\min(u_c, |u_y|)} f(n) \left(\exp \int_{\hat{n}}^{\bar{n}} \frac{u_{yn}}{u_y} d\nu \right) dn \tag{6.33}$$

for any $\bar{n}$. From (6.33) and (6.29), one concludes that $\varphi(\bar{n}) > 0$ for $\bar{n} > \hat{n}$ sufficiently close to $\hat{n}$. This is incompatible with the first-order condition (6.8) for the slack variable $s(\bar{n})$. ∎

Given Lemma 6.7, the following lemma shows that downward incentive compatibility is everywhere locally binding. This is the analogue of Lemma 5.8 in the finite case.

Lemma 6.8 *For any $n \in (n^0, n^1)$, $s(n) = 0$ and*

$$v'(n) = u_n(c(n), y(n), n). \tag{6.34}$$

Proof. Suppose that $s(\hat{n}) > 0$ for some $\hat{n} \in (n^0, n^1)$. By the complementary slackness condition for $s(\hat{n})$, it follows that $\varphi(\hat{n}) = 0$. By Lemma 6.7, it follows that $\psi(\hat{n}) < 0$. Because ψ is continuous, one must have $\psi(n) < 0$ for all n in some neighbourhood $[\hat{n} - \delta, \hat{n} + \delta]$ of $\hat{n}$. By the complementary slackness condition for $q(n)$, it follows that $q(n) = 0$ for all n in this neighbourhood. By Lemma 6.3, therefore, one also has $\frac{dy}{dn}(n) = 0$ for all n in this neighbourhood. By (6.3) and (2.6) it follows that $s(n) = 0$ for all $n \in [\hat{n} - \delta, \hat{n} + \delta]$. The assumption that $s(\hat{n}) > 0$ must therefore be false. ∎

Proceeding by the same arguments as in the finite case, one now obtains:

Corollary 6.9 *The functions $c(\cdot)$ and $y(\cdot)$ are nondecreasing and co-monotonic.*

Corollary 6.10 *The allocation $(c(\cdot), y(\cdot))$ is incentive-compatible.*

Lemma 6.11 *The function $y(\cdot) - c(\cdot)$ is nondecreasing and co-monotonic with $c(\cdot)$ and $y(\cdot)$.*

Lemma 6.12 *For $n \in N$, consumption is bounded away from zero.*

Lemma 6.13 *On any compact subset of (n^0, n^1) on which $y(n) > 0$, $(c(n), y(n))$ is distorted downward and bounded away from efficiency.*

Proof. If the lemma is false, then, by continuity, there exists $\hat{n} \in (n^0, n^1)$ such that $y(\hat{n}) > 0$ and

$$(c(\hat{n}), y(\hat{n})) = (c^*(\hat{n}), y^*(\hat{n})). \tag{6.35}$$

By Lemma 6.12, one has $c(\hat{n}) > 0$. Since also $y^*(\hat{n}) > 0$, the same argument as in the proof of Lemma 6.5 implies that $c(n) < c(\hat{n})$ for $n < \hat{n}$. Therefore, $\psi(\hat{n}) = 0$.

Since $\psi(\hat{n}) = 0$ and, by (6.7), $\psi(n) \leq 0$ for $n > \hat{n}$, there exists a sequence $\{n^k\}$ which converges to $\hat{n}$ from above such that $\psi'(n^k) \leq 0$ for all k. By the monotonicity of the allocation, one has $(c(n^k), y(n^k)) \gg (0, 0)$ for all k, so, for $n = n^k$, (6.16) must hold as an equation. Thus, $\psi'(n^k) \leq 0$ implies

$$\lambda \frac{u_c^k + u_y^k}{u_y^k} f(n^k) - \varphi(n^k)(u_{nc}^k - \frac{u_c^k}{u_y^k} u_{ny}^k) \leq 0, \tag{6.36}$$

where the derivatives u_c^k, u_y^k, etc. are all evaluated at $(c(n^k), y(n^k), n^k)$. Because the allocation, as well as the density f and the costate variable φ, are continuous, one can take limits in (6.36), to obtain

$$\lambda \frac{u_c + u_y}{u_y} f(\hat{n}) - \varphi(\hat{n})(u_{nc} - \frac{u_c}{u_y} u_{ny}) \leq 0, \tag{6.37}$$

where u_c, u_k, etc. are evaluated at $(c(\hat{n}), y(\hat{n}), \hat{n})$. Because $(c(\hat{n}), y(\hat{n})) = (c^*(\hat{n}), y^*(\hat{n}))$ is strictly positive, the first term on the left-hand side is zero. By SSCC, $(u_{nc} - \frac{u_c}{u_y} u_{ny})$ is strictly positive. Therefore, (6.37) implies $\varphi(\hat{n}) \geq 0$. By (6.8), it follows that $\varphi(\hat{n}) = 0$.

By Lemma 6.7, for $\hat{n} \in (n^0, n^1)$, $\varphi(\hat{n}) = \psi(\hat{n}) = 0$ implies $c(\hat{n}) = 0$, contrary to Lemma 6.12. The assumption that the lemma is false has thus led to a contradiction. ■

The preceding results show that any solution to the weakly relaxed income tax problem exhibits Properties A, B, and D, as specified in Section 3: Property A holds by Corollary 6.6, Property B by Lemma 6.13, Property D by Lemmas 6.11 and 6.12 and Corollary 6.6. The following lemma shows that Property C is also satisfied.

Lemma 6.14 *If $c(\cdot)$ and $y(\cdot)$ are strictly increasing at n^0, then*

$$(c(n^0), y(n^0)) = (c^*(n^0, v(n^0)), y^*(n^0, v(n^0))); \tag{6.38}$$

if the monotonicity constraint on $c(\cdot)$ is strictly binding at n_0 and if $y(n^0) > 0$, then

$$(c(n^0), y(n^0)) \ll (c^*(n^0, v(n^0)), y^*(n^0, v(n^0))). \tag{6.39}$$

Proof. If $c(\cdot)$ and $y(\cdot)$ are strictly increasing at n^0, one has $c(n) > 0$ and $y(n) > 0$ for $n > n^0$. Then (6.14) and (6.15) hold as equations. Because φ, f, and the allocation $(c(\cdot), y(\cdot))$ are continuous, it follows that ψ' and φ' are also continuous. Therefore, (6.14) and (6.15) hold as equations for $n = n^0$, as well as $n > n^0$, and so does (6.16). By the transversality condition (6.11), this yields

$$\psi'(n^0) = \lambda \frac{u_c + u_y}{u_y} f(n^0), \tag{6.40}$$

where u_c and u_y are evaluated at $(c(n^0), y(n^0), n^0)$. If $c(\cdot)$ and $y(\cdot)$ are strictly increasing at n^0, there exists a sequence $\{n^k\}$ converging to n^0 from above such that $\psi(n^k) = 0$ for all k. Because ψ' is continuous, it follows that there also exists a sequence $\{n^\ell\}$ converging to n^0 from above such that $\psi'(n^\ell) = 0$ for all ℓ. By continuity, therefore, $\psi'(n^0) = 0$, so (6.40) implies (6.38). By contrast, if $\psi'(n^0) < 0$ and $y(n^0) > 0$, (6.40) is again valid, but implies (6.39). ■

Theorem 6.1 follows because, by the argument at the end of Section 4, Corollary 6.10 implies that a piecewise continuously differentiable allocation $(c(\cdot), y(\cdot))$ solves the optimal income tax problem if and only if it also solves the weakly relaxed income tax problem.

A Proof of Proposition 2.1

To simplify things for the reader, I begin by restating condition DR and Proposition 2.1.

DR **Desirability of Redistribution:** For any $(c, y, n) \in \Re^2_+ \times [n^0, n^1)$, there exists $\varepsilon > 0$ such that $n + \varepsilon \in N$, and, for all $n' \in (n, n + \varepsilon]$ and all $(c', y') \in \Re^2_+$, the following hold:

(a) if $c' = c$ and $y' < y$, then

$$|u_y(c, y, n)| > |u_y(c', y', n')|\,; \tag{A.1}$$

(b) if $c' > c$, $u(c', y', n') \geq u(c, y, n')$, and if, moreover, (c, y) is efficient for n, and (c', y') is efficient or distorted downwards from efficiency for n', then

$$|u_y(c, y, n)| > \min(u_c(c', y', n'), |u_y(c', y', n')|) \;\text{ if }\; y > 0, \tag{A.2}$$

and

$$u_c(c, y, n) > \min(u_c(c', y', n'), |u_y(c', y', n')|) \;\text{ if }\; y = 0. \tag{A.3}$$

Proposition A.1 *Assume RC, ND, and SSCC, and suppose that, for any n, the indifference curves of the utility function $u(\cdot, \cdot, n)$ have strictly positive Gaussian curvature.*[17] *Assume also that N is an interval. Then condition DR holds if u is concave in c and y, and, moreover,*

$$u_{cn}(c, 0, n) \leq 0, \;\; u_{cc}(c, 0, n) < 0, \tag{A.4}$$

$$u_{yn}(c, y, n) > 0 \tag{A.5}$$

for all (c, y, n), and

$$u_{ny}(c^*(n, v), y^*(n, v), n)\frac{\partial y^*}{\partial v} + u_{nc}(c^*(n, v), y^*(n, v), n)\frac{\partial c^*}{\partial v} < 0 \tag{A.6}$$

for all $n \in N$ and all v in the range of $u(\cdot, \cdot, n)$. Under these assumptions on the functions u, c^*, and y^*, condition DR also holds if N is a finite set and the differences between neighbouring elements of N are uniformly small.

The proof of this proposition proceeds in several steps. The first step concerns part (a) of condition DR.

[17] I.e. that the quadratic form $u_y^2 u_{cc} - 2u_c u_y u_{cy} + u_c^2 u_{yy}$ is everywhere strictly negative.

Lemma A.2 *If* $(c, y, n) \in \Re_+^2 \times [n^0, n^1)$ *and* $(c', y', n') \in \Re_+^2 \times (n, n_1]$ *are such that* $c' = c$ *and* $y' \leq y$*, then*

$$\left|u_y(c', y', n')\right| < |u_y(c, y, n)|\,. \tag{A.7}$$

Proof. Immediate from (A.5) and the concavity of u. ■

The next two steps concern the validity of part (b) of condition DR when the outcome pair (c', y') for type n' lies on the indifference curve $I(n')$ of type n' through the outcome pair (c, y) for type $n < n'$; see Figure 1 in the text.

The first of these two steps concerns the case where (c, y) satisfies the first-order condition for efficiency as an equation.

Lemma A.3 *If* $n \in [n^0, n^1)$ *and* $(c, y) \in \Re_+^2$ *are such that*

$$u_c(c, y, n) + u_y(c, y, n) = 0, \tag{A.8}$$

then there exists $\varepsilon > 0$ *such that, for any* $n' \in (n, n + \varepsilon]$ *and any* $(c', y') \in \Re_+^2$ *that satisfies*

$$u(c', y', n') = u(c, y, n') \tag{A.9}$$

and $c \leq c' \leq c^*(n', u(c, y, n'))$*, one has*

$$\left|u_y(c', y', n')\right| < |u_y(c, y, n)|\,. \tag{A.10}$$

Proof. If (c, y) satisfies (A.8), then, for $n' > n$, SSCC implies

$$u_c(c, y, n') + u_y(c, y, n') > 0.$$

By RC, there exists a unique pair $(c^*(n', u(c, y, n')), y^*(n', u(c, y, n')))$ that lies on $I(n')$ and is efficient for n'. This pair is strictly greater than (c, y) and satisfies the first-order condition for an interior efficient point, $u_c + u_y = 0$.

By RC, the indifference curve $I(n')$ of type n' through (c, y) is strictly convex. Its slope is strictly increasing. For any point (c', y') on $I(n')$ that lies between the efficient pair $(c^*(n', u(c, y, n')), y^*(n', u(c, y, n')))$ and the reference point (c, y), there must therefore exist some $\delta \in [0, 1]$ such that

$$\frac{|u_y(c', y', n')|}{u_c(c', y', n')} = \delta \frac{|u_y(c, y, n')|}{u_c(c, y, n')} + (1 - \delta); \tag{A.11}$$

in (A.26), the left-hand side indicates the slope of $I(n')$ at (c', y'), the fraction $\frac{|u_y(c,y,n')|}{u_c(c,y,n')}$ on the right-hand side the slope of $I(n')$ at (c, y); the slope of $I(n')$ at the efficient point $(c^*(n', u(c, y, n')), y^*(n', u(c, y, n')))$ is of course one.

Conversely, for the given n' and any $\delta \in [0,1]$, there exists a pair $(\hat{c}(n',\delta), \hat{y}(n',\delta))$ such that $(c',y') = (\hat{c}(n',\delta), \hat{y}(n',\delta))$ lies between (c,y) and $(c^*(n'), y^*(n'))$ and is the unique solution to equations (A.9) and (A.11). By construction, one has $(\hat{c}(n,\delta), \hat{y}(n,\delta)) = (c,y)$ for all δ. One also has $(\hat{c}(n',0), \hat{y}(n',0)) = (c^*(n', u(c,y,n')), y^*(n', u(c,y,n')))$ for all n'.

By standard arguments, relying on the implicit function theorem, the assumption that indifference curves have positive Gaussian curvature implies that the functions $\hat{c}(\cdot,\cdot)$ and $\hat{y}(\cdot,\cdot)$ that are defined by (A.9) and (A.11) are continuously differentiable. It follows that, for any $\delta \in [0,1]$ and any $n' \geq n$, the derivative

$$\frac{du_y(\hat{c}(n',\delta), \hat{y}(n',\delta), n')}{dn'} = \left[u_{yc}\frac{\partial \hat{c}}{\partial n'}(n',\delta) + u_{yy}\frac{\partial \hat{y}}{\partial n'}(n',\delta) + u_{yn}\right] \quad \text{(A.12)}$$

is well defined. At $n' = n$ and any δ, one computes[18]

$$\frac{\partial \hat{c}}{\partial n'}(n,\delta) = \frac{\partial \hat{y}}{\partial n'}(n,\delta) = -(1-\delta)\frac{(u_{cn} + u_{yn})}{u_{cc} + u_{cy} + u_{yc} + u_{yy}}. \quad \text{(A.13)}$$

For any $\delta \in [0,1]$, one therefore obtains

$$\begin{aligned}
&\frac{du_y(\hat{c}(n',\delta), \hat{y}(n',\delta), n')}{dn'}(n,\delta) \\
= \;& -(1-\delta)\frac{(u_{cn} + u_{yn})}{u_{cc} + u_{cy} + u_{yc} + u_{yy}}\,(u_{yc} + u_{yy}) + u_{yn} \\
= \;& (1-\delta)\left[-\frac{(u_{yc} + u_{yy})}{u_{cc} + u_{cy} + u_{yc} + u_{yy}}u_{cn} + \frac{(u_{cc} + u_{cy})}{u_{cc} + u_{cy} + u_{yc} + u_{yy}}u_{yn}\right] + \delta u_{yn} \\
= \;& (1-\delta)\left[u_{cn}\frac{\partial c^*(n,v)}{\partial v} + u_{yn}\frac{\partial y^*(n,v)}{\partial v}\right] + \delta u_{yn} \\
\geq \;& \min\left[-\left(u_{cn}\frac{\partial c^*(n,v)}{\partial v} + u_{yn}\frac{\partial y^*(n,v)}{\partial v}(n)\right), u_{yn}\right]. \qquad \text{(A.14)}
\end{aligned}$$

The right-hand side of (A.14) is independent of δ. By (A.5) and (A.6), it is also strictly positive. It follows that, for some $\varepsilon > 0$, $n' \in (n, n+\varepsilon]$ implies

$$\frac{du_y(\hat{c}(n',\delta), \hat{y}(n',\delta), n')}{dn'}(n',\delta) > 0$$

for all $\delta \in [0,1]$. The claim follows immediately. ■

The next step concerns the case where (c,y) satisfies the first-order condition for efficiency as a strict inequality.

[18] Positivity of the Gaussian curvature implies that the denominator in (A.13) is strictly negative.

Lemma A.4 *Let $n \in [n^0, n^1)$ and $(c, y) \in \Re^2_+$ be such that (c, y) is efficient for n, with*

$$u_c(c, y, n) + u_y(c, y, n) < 0; \tag{A.15}$$

then there exists $\varepsilon > 0$ such that, for any $n' \in (n, n + \varepsilon]$ and any $(c', y') \in \Re^2_+$ that satisfies

$$u(c', y', n') = u(c, y, n') \tag{A.16}$$

and $c < c' \leq c^*(n', u(c, y, n'))$, one has

$$\left|u_y(c', y', n')\right| < |u_y(c, y, n)| \tag{A.17}$$

and

$$u_c(c', y', n') < u_c(c, y, n) \quad \text{if} \quad y = 0. \tag{A.18}$$

Proof. If (c, y) satisfies (A.15), then by RC, one also has

$$u_c(c, y, n') + u_y(c, y, n') < 0 \tag{A.19}$$

for n' sufficiently close to n. If (c, y) satisfying (A.15) is efficient for n, one must have $c = 0$ or $y = 0$. Such (c, y) satisfying (A.19) is then also efficient for n', and, for (c', y') satisfying (A.16) and $c \leq c' \leq c^*(n', u(c, y, n'))$, one has $(c', y') = (c, y)$. Then (A.17) and (A.18) follow from (A.5) and (A.4) by standard calculus. ■

The final step concerns those outcome pairs (c', y') that type n' prefers to the reference pair (c, y), that have consumption greater than c and that are efficient or distorted downwards from efficiency for type n'. In Figure 1, the set of these outcome pairs is represented by the shaded area to the left of the curve $A - A$ of efficient points for n', strictly above the indifference curve $I(n')$ and strictly above the horizontal line through the reference point (c, y). The argument is adapted from Brunner (1995).

Lemma A.5 *Let $n \in [n^0, n^1)$ and $(c, y) \in \Re^2_+$ be such that (c, y) is efficient for n, and let ε be given by Lemma A.3 or A.4. Then for any $n' \in (n, n + \varepsilon]$ and any $(c', y') \in \Re^2_+$ that is efficient or distorted downwards from efficiency for n' and satisfies that $c' > c$ and $u(c', y', n') > u(c, y, n')$, one has*

$$\left|u_y(c', y', n')\right| < |u_y(c, y, n)| \tag{A.20}$$

and

$$\min(u_c(c', y', n'), \left|u_y(c', y', n')\right|) < u_c(c, y, n) \quad \text{if} \quad y = 0. \tag{A.21}$$

Proof. Fix n, (c,y), n', (c',y') as specified in the lemma. Define a shadow price q for the outcome (c',y') for n' by setting

$$qu_c(c',y',n') + u_y(c',y',n') = 0 \tag{A.22}$$

if

$$u_c(c',y',n') + u_y(c',y',n') \geq 0, \tag{A.23}$$

and

$$q = 1 \tag{A.24}$$

if

$$u_c(c',y',n') + u_y(c',y',n') < 0. \tag{A.25}$$

RC implies that $q \leq 1$. Given that $q \leq 1$, RC also implies that, for any v in the range of u, the problem of minimizing $c'' - qy''$ under the constraint that $u(c'',y'',n') \geq v$ has a unique solution. Let $(\bar{c}(v), \bar{y}(v))$ be this solution. Again by RC, one has $(\bar{c}(v), \bar{y}(v)) \leq (c^*(n',v), y^*(n',v))$ for all v.

From (A.22) - (A.25), one easily finds that $(c',y') = (\bar{c}(u(c',y',n')), \bar{y}(u(c',y',n')))$. Because u is concave, one also has

$$\left|u_y(\bar{c}(v), \bar{y}(v), n')\right| \geq \left|u_y(c',y',n')\right| \tag{A.26}$$

and

$$u_c(\bar{c}(v), \bar{y}(v), n') \geq u_c(c',y',n') \tag{A.27}$$

whenever $v < u(c',y',n')$. In particular,

$$\left|u_y(\bar{c}(u(c,y,n')), \bar{y}(u(c,y,n')), n')\right| \geq \left|u_y(c',y',n')\right| \tag{A.28}$$

and

$$u_c(\bar{c}(u(c,y,n')), \bar{y}(u(c,y,n')), n') \geq u_c(c',y',n'). \tag{A.29}$$

If $\bar{c}(u(c,y,n')) \geq c$, (A.20) follows from (A.28) because, by Lemma A.3 or A.4, one has $|u_y(\bar{c}(u(c,y,n')), \bar{y}(u(c,y,n')), n')| < |u_y(c,y,n)|$.

If $\bar{c}(u(c,y,n')) < c$, then, by the intermediate value theorem, there exists $\hat{v} \in (u(c,y,n'), u(c',y',n'))$ such that $\bar{c}(\hat{v}) = c$ and $\bar{y}(v) < y$. For this $\hat{v}$, Lemma A.2 yields $|u_y(\bar{c}(\hat{v}), \bar{y}(\hat{v}), n')| < |u_y(c,y,n)|$, so (A.20) follows from (A.26).

As for (A.21), I first note that, trivially, one must have $\bar{c}(u(c,0,n')) \geq c$. Therefore, if $y = 0$ and $u_c(c,y,n) + u_y(c,y,n) < 0$, Lemma A.4 yields $u_c(\bar{c}(u(c,y,n')), \bar{y}(u(c,y,n')), n') < u_c(c,y,n)$. If one combines this inequality with (A.29), one obtains (A.21). If $u_c(c,y,n) + u_y(c,y,n) = 0$, (A.21) follows directly from (A.20). ■

Upon combining Lemmas A.3 - A.5, one finds that, for n, c, y and n', c', y' as specified in these lemmas, (A.2) follows from (A.10), (A.17), or (A.20). If $u_c(c, y, n) = |u_y(c, y, n)|$, (A.3) follows immediately. If $y = 0$ and $u_c(c, y, n) < |u_y(c, y, n)|$, (A.3) follows from (A.18) or (A.21). To complete the proof of Proposition A.1, it suffices to note that, if N is an interval, then there is no loss of generality in assuming that $n + \varepsilon$ is an element of N. The same is true if N is a finite set and the elements of N are sufficiently close.

B Proof of Lemma 6.2

Lemma B.1 *An allocation $(c(\cdot), y(\cdot))$ with nondecreasing $c(\cdot)$ is downward incentive compatible if and only if the indirect utility function $v(\cdot) = u(c(\cdot), y(\cdot), \cdot)$ satisfies*

$$v(n) = S(n) + \int_{n^0}^{n} u_n(c(n'), y(n'), n')dn' \tag{B.1}$$

for some nondecreasing function $S(\cdot)$. If the allocation is piecewise continuously differentiable, the functions $v(\cdot)$ and $S(\cdot)$ are also piecewiese continuously differentiable, and one has

$$v'(n) \geq u_n(c(n), y(n), n). \tag{B.2}$$

Proof. The argument follows Mirrlees (1976). For any n, let $S(n)$ be the difference between $v(n)$ and the integral $\int_{n^0}^{n} u_n(c(n'), y(n'), n')dn'$. If $S(\cdot)$ is a nondecreasing function, one has

$$\int_{n'}^{n} \chi(n'', n)dS(n'') \geq 0 \tag{B.3}$$

for all n, all $n' < n$, and every nonnegative-valued function χ. By the definition of $S(\cdot)$, (B.3) is equivalent to the inequality

$$\int_{n'}^{n} \chi(n'', n)[u_c(c(n''), y(n''), n'')dc(n'') + u_y(c(n''), y(n''), n'')dy(n'')] \geq 0.$$

With $\chi(n'', n) = \frac{u_y(c(n''), y(n''), n)}{u_y(c(n''), y(n''), n'')}$, it follows that

$$\int_{n'}^{n} u_y(c(n''), y(n''), n)dy(n'') + \int_{n'}^{n} \frac{u_y(c(n''), y(n''), n)}{u_y(c(n''), y(n''), n'')} u_c(c(n''), y(n''), n'')dc(n'') \geq 0 \tag{B.4}$$

for all n and all $n' < n$.

By SSCC and RMC, one also has

$$\frac{-u_y(c(n''), y(n''), n)}{u_c(c(n''), y(n''), n)} < \frac{-u_y(c(n''), y(n''), n'')}{u_c(c(n''), y(n''), n'')},$$

whenever $n'' < n$. Because $c(\cdot)$ is nondecreasing, it follows that

$$\begin{aligned} -\int_{n'}^{n} \frac{u_y(c(n''), y(n''), n)}{u_y(c(n''), y(n''), n'')} u_c(c(n''), y(n''), n'') dc(n'') \\ \geq -\int_{n'}^{n} u_c(c(n''), y(n''), n) dc(n'') \end{aligned} \tag{B.5}$$

for all n and all $n' < n$. Now (B.4) and (B.5) imply

$$\int_{n'}^{n} u_c(c(n''), y(n''), n) dc(n'') + \int_{n'}^{n} u_y(c(n''), y(n''), n) dy(n'') \geq 0, \tag{B.6}$$

hence

$$u(c(n), y(n), n) \geq u(c(n'), y(n'), n) \tag{B.7}$$

for all n and all $n' < n$. Monotonicity of $c(\cdot)$ and $S(\cdot)$ is thus sufficient for downward incentive compatibility.

Conversely, for all n and all $n' < n$, downward incentive compatibility implies

$$v(n) - v(n') \geq \int_{n'}^{n} u_n(c(n'), y(n'), n'') dn''. \tag{B.8}$$

By standard arguments, it follows that

$$\begin{aligned} v(n) - v(n') &= \sum_{k=1}^{K} (v(n_{k+1}) - v(n_k)) \\ &\geq \sum_{k=1}^{K} \int_{n_k}^{n_{k+1}} u_n(c(n_k), y(n_k), n'') dn'' \end{aligned} \tag{B.9}$$

for every increasing sequence $\{n_k\}_{k=1}^{K}$ with $n_1 = n'$ and $n_K = n$. Upon taking limits across sequences $\{n_k\}_{k=1}^{K}$ as K goes out of bounds and the sequence $\{n_k\}_{k=1}^{K}$ becomes dense in the interval $[n', n]$, one concludes that

$$v(n) - v(n') \geq \int_{n'}^{n} u_n(c(n''), y(n''), n'') dn''$$

and, hence, that $S(n) \geq S(n')$ for all n and all $n' < n$.

If $c(\cdot)$ and $y(\cdot)$ are piecewise continuously differentiable, with derivatives $q(\cdot)$ and $z(\cdot)$, one can use the equation $v(n) = u(c(n), y(n), n)$ to show that $v(\cdot)$ is also piecewise continuously differentiable, with derivative

$$v'(n) = u_c q(n) + u_y z(n) + u_n. \tag{B.10}$$

Equation (B.1) then yields

$$S(n) - S(n^0) = \int_{n^0}^{n} [u_c q(n') + u_y z(n')] dn'. \tag{B.11}$$

Thus function $S(\cdot)$ is also piecewise continuously differentiable, with derivative $u_c q + u_y z \geq 0$. (B.2) follows immediately. ■

C Additional Proofs for Section 6.2

In this appendix, I show that the Lagrange multiplier for the feasibility constraint is positive, and I provide the full proof of Lemma 6.7 in the text. Both proofs are complicated by the appearance of the term $\psi'(n)$ in the conditions for consumption. To deal with these complications, both proofs rely on the following lemma.

Lemma C.1 *For any $\hat{n}, \bar{n} \in [n^0, n^1]$, $\psi(\hat{n}) = 0$ and $\psi(\bar{n}) \leq 0$ imply*

$$\int_{\bar{n}}^{\hat{n}} \frac{\psi'(n)}{u_c} f(n) \exp\left(\int_{n^0}^{n} \frac{u_{nc}}{u_c} d\nu\right) dn \geq 0, \tag{C.1}$$

with equality if and only if $\psi(\bar{n}) = 0$.

Proof. First, let $\bar{n} < \hat{n}$. Neglecting null sets, one may suppose that $\psi'(n)$ is nonzero only on intervals of constancy of $c(\cdot)$. On such intervals, by Lemma 6.3 in the text, $y(\cdot)$ is also constant. Let $\mathcal{I}$ be the collection of such intervals between $\bar{n}$ and $\hat{n}$ and note that $\mathcal{I}$ is at most countable. For any $I \in \mathcal{I}$, let $n_0(I)$ and $n_1(I)$ be the infimum and the supremum of I, and let $c(I), y(I)$ be the common value of $(c(n'), y(n'))$ on $(n_0(I), n_1(I))$. Because, outside the union of intervals in $\mathcal{I}$, $\psi'(n)$ must vanish for almost all n, one can write:

$$\int_{\bar{n}}^{\hat{n}} \frac{\psi'(n)}{u_c} f(n) \exp\left(\int_{n^0}^{n} \frac{u_{nc}}{u_c} d\nu\right) dn = \sum_{I \in \mathcal{I}} \int_{n_0(I)}^{n_1(I)} \frac{\psi'(n)}{u_c} \exp\left(\int_{n^0}^{n} \frac{u_{nc}}{u_c} d\nu\right) dn. \tag{C.2}$$

For any $I \in \mathcal{I}$ and any $n' \in (n_0(I), n_1(I))$, one has $(c(n'), y(n')) = (c(I), y(I))$. Therefore

$$\begin{aligned}
\exp\left(\int_{n^0}^{n} \frac{u_{nc}}{u_c} d\nu\right) &= \exp\left(\int_{n^0}^{n_0(I)} \frac{u_{nc}}{u_c} d\nu + \int_{n_0(I)}^{n} \frac{d \ln u_c}{dn'} dn'\right) \\
&= \exp\left(\int_{n^0}^{n_0(I)} \frac{u_{nc}}{u_c} d\nu + \ln \frac{u_c(c(I), y(I), n)}{u_c(c(I), y(I), n_0(I))}\right) \\
&= \frac{u_c(c(I), y(I), n)}{u_c(c(I), y(I), n_0(I))} \exp\left(\int_{n^0}^{n_0(I)} \frac{u_{nc}}{u_c} d\nu\right).
\end{aligned}$$

For any $I \in \mathcal{I}(n)$, it follows that

$$\begin{aligned}
&\int_{n_0(I)}^{n_1(I)} \frac{\psi'(n)}{u_c(c(n), y(n), n)} \exp\left(\int_{n^0}^{n} \frac{u_{nc}}{u_c} d\nu\right) dn \\
= \;& \frac{1}{u_c(c(I), y(I), n_0(I))} \exp\left(\int_{n^0}^{n_0(I)} \frac{u_{nc}}{u_c} d\nu\right) \int_{n_0(I)}^{n_1(I)} \psi'(n) dn \\
= \;& \frac{1}{u_c(c(I), y(I), n_0(I))} \exp\left(\int_{n^0}^{n_0(I)} \frac{u_{nc}}{u_c} d\nu\right) [\psi(n_1(I)) - \psi(n_0(I))].
\end{aligned}$$

If $\psi(\bar{n}) = 0$, then, by the definition of $n_0(I)$ and $n_1(I)$ and the continuity of ψ, one has $\psi(n_0(I)) = \psi(n_1(I)) = 0$, hence

$$\int_{n_0(I)}^{n_1(I)} \frac{\psi'(n)}{u_c} \exp\left(\int_{n^0}^{n} \frac{u_{nc}}{u_c} d\nu\right) dn = 0 \tag{C.3}$$

for all $I \in \mathcal{I}$. Given that the set $\mathcal{I}$ is at most countable, (C.2) and (C.3) imply

$$\int_{\bar{n}}^{\hat{n}} \frac{\psi'(n)}{u_c} f(n) \exp\left(\int_{n^0}^{n} \frac{u_{nc}}{u_c} d\nu\right) dn = 0.$$

If $\psi(\bar{n}) < 0$, there exists an initial interval I_1 of constancy of $c(\cdot)$ such that $\bar{n} = n_0(I_1)$. For $I \in \mathcal{I} \backslash \{I_1\}$, one again has $\psi(n_0(I)) = \psi(n_1(I)) = 0$, so (C.3) holds and (C.2) - (C.3) imply

$$\begin{aligned}
&\int_{\bar{n}}^{\hat{n}} \frac{\psi'(n)}{u_c} f(n) \exp\left(\int_{n^0}^{n} \frac{u_{nc}}{u_c} d\nu\right) dn \\
= \;& \int_{\bar{n}}^{n_1(I_1)} \frac{\psi'(n)}{u_c} f(n) \exp\left(\int_{n^0}^{n} \frac{u_{nc}}{u_c} d\nu\right) dn \\
= \;& \frac{1}{u_c(c(I), y(I), \bar{n})} \exp\left(\int_{n^0}^{\bar{n}} \frac{u_{nc}}{u_c} d\nu\right) [\psi(n_1(I_1)) - \psi(\bar{n})].
\end{aligned}$$

By the definition of $n_1(I_1)$ and the continuity of ψ, one has $\psi(n_1(I_1)) = 0$. Therefore $\psi(\bar{n}) < 0$ implies

$$\int_{\bar{n}}^{\hat{n}} \frac{\psi'(n)}{u_c} f(n) \exp\left(\int_{n^0}^{n} \frac{u_{nc}}{u_c} d\nu\right) dn > 0. \tag{C.4}$$

If $\bar{n} > \hat{n}$, a precisely symmetric argument shows that

$$\int_{\hat{n}}^{\bar{n}} \frac{\psi'(n)}{u_c} f(n) \exp\left(\int_{n^0}^{n} \frac{u_{nc}}{u_c} d\nu\right) dn = 0 \tag{C.5}$$

if $\psi(\bar{n}) = 0$ and

$$\int_{\hat{n}}^{\bar{n}} \frac{\psi'(n)}{u_c} f(n) \exp\left(\int_{n^0}^{n} \frac{u_{nc}}{u_c} d\nu\right) dn < 0 \tag{C.6}$$

if $\psi(\bar{n}) < 0$. (C.4) follows immediately from (C.6). ■

Lemma C.2 $\lambda > 0$.

Proof. If the lemma is false, $\lambda = 0$. Then condition (6.14) in the text, the condition for the costate variable $\psi(\cdot)$, can be written as

$$\frac{\psi'(n)}{u_c} + f(n) + \varphi'(n) + \varphi(n)\frac{u_{nc}}{u_c} \leq 0, \tag{C.7}$$

which in turn implies

$$\frac{\psi'(n)}{u_c} \exp\left(\int_{n^0}^{n} \frac{u_{nc}}{u_c} d\nu\right) + \frac{d}{dn}\left(\varphi(n) \exp\left(\int_{n^0}^{n} \frac{u_{nc}}{u_c} d\nu\right)\right) < 0.$$

By integration, one obtains

$$\int_{n^0}^{n^1} \frac{\psi'(n)}{u_c} \exp\left(\int_{n^0}^{n} \frac{u_{nc}}{u_c} d\nu\right) dn + \varphi(n_1) \exp\left(\int_{n^0}^{n} \frac{u_{nc}}{u_c} d\nu\right) - \varphi(n^0) < 0.$$

By the transversality condition (6.11), therefore,

$$\int_{n^0}^{n^1} \frac{\psi'(n)}{u_c} \exp\left(\int_{n^0}^{n} \frac{u_{nc}}{u_c} d\nu\right) dn < 0. \tag{C.9}$$

However, by Lemma C.1, in combination with (6.7) and the transversality condition (6.12), one also has

$$\int_{n^0}^{n^1} \frac{\psi'(n)}{u_c} \exp\left(\int_{n^0}^{n} \frac{u_{nc}}{u_c} d\nu\right) dn \geq 0. \tag{C.10}$$

The assumption that $\lambda = 0$ has thus led to a contradiction and must be false. ■

Lemma C.3 *For any* $n \in (n^0, n^1)$, $c(n) > 0$ *and* $\varphi(n) = \psi(n) = 0$ *imply*

$$u_c(c(n), y(n), n) \leq \lambda \tag{C.11}$$

and

$$|u_y(c(n), y(n), n)| \leq \lambda \quad \textit{if } y(n) > 0. \tag{C.12}$$

Proof. Let $\hat{n} \in (n^0, n^1)$ be such that $c(\hat{n}) > 0$ and $\varphi(\hat{n}) = \psi(\hat{n}) = 0$. I first prove that

$$\lambda \geq u_c(c(\hat{n}), y(\hat{n}), \hat{n}). \tag{C.13}$$

If, for some $\bar{n} < \hat{n}$, one has $\psi(n) < 0$ for $n \in (\bar{n}, \hat{n})$, then there is bunching, and one has $c(n) = c(\hat{n}) > 0$ for $n \in [\bar{n}, \hat{n}]$. For $n \in [\bar{n}, \hat{n}]$, therefore, condition (6.14) in the text must hold as an equation. Thus

$$\psi'(n) + \varphi'(n)u_c + \varphi(n)u_{cn} = (\lambda - u_c)f(n) \tag{C.14}$$

for all $n \in [\bar{n}, \hat{n}]$, where u_c and u_{cn} are evaluated at $(c(n), y(n), n)$. Because $c(n)$ is constant on $[\bar{n}, \hat{n}]$, Lemma 6.3 implies that $y(n)$ is also constant on $[\bar{n}, \hat{n}]$. Therefore, one has

$$\frac{d}{dn}u_c(c(n), y(n), n) = u_{cn}(c(n), y(n), n) \tag{C.15}$$

for $n \in [\bar{n}, \hat{n}]$. On the interval $[\bar{n}, \hat{n}]$, equation (C.14) thus can be rewritten as

$$\psi'(n) + \frac{d}{dn}\left[\varphi(n)u_c(c(n), y(n), n)\right] = (\lambda - u_c)f(n). \tag{C.16}$$

Because $\psi(n)$ and $\varphi(n)$ are nonpositive for all n and because $\varphi(\hat{n}) = \psi(\hat{n}) = 0$, one has

$$\psi(n) + \varphi(n)u_c(c(n), y(n), n) \leq \psi(\hat{n}) + \varphi(\hat{n})u_c(c(\hat{n}), y(\hat{n}), \hat{n}) \tag{C.17}$$

for all $n \in [\bar{n}, \hat{n}] = 0$. Therefore, it must be the case that

$$\psi'(n) + \frac{d}{dn}\left[\varphi(n)u_c(c(n), y(n), n)\right] \geq 0 \tag{C.18}$$

and, by (C.16), that

$$(\lambda - u_c)f(n) \geq 0 \tag{C.19}$$

for n between $\bar{n}$ and $\hat{n}$, arbitrarily close to $\hat{n}$. By the continuity of the allocation, (C.13) follows.

If, instead, there exists a sequence $\{n^k\}$ converging to $\hat{n}$ from below such that $\psi(n^k) = 0$ for all k, there need not be any bunching at $c(\hat{n})$, but, by the continuity of $c(\cdot)$, one still has $c(n) > 0$, and one may suppose that equation (C.14) must hold for n close to $\hat{n}$. This equation can be rewritten as:

$$\frac{d}{dn}\left(\varphi(n)\exp\left(\int_{n^0}^{n}\frac{u_{nc}}{u_c}d\nu\right)\right) = \left[\frac{\lambda - u_c}{u_c}f(n) - \frac{\psi'(n)}{u_c}\right]\exp\left(\int_{n^0}^{n}\frac{u_{nc}}{u_c}d\nu\right). \tag{C.20}$$

Integration between n^k and $\hat{n}$ yields:

$$\begin{aligned}\varphi(\hat{n})\exp\left(\int_{n^0}^{n}\frac{u_{nc}}{u_c}d\nu\right) &= \varphi(n^k)\exp\left(\int_{n^0}^{n^k}\frac{u_{nc}}{u_c}d\nu\right) \\ &+ \int_{n^K}^{\hat{n}}\frac{\lambda - u_c}{u_c}f(n)\exp\left(\int_{n^0}^{n}\frac{u_{nc}}{u_c}d\nu\right)dn \\ &- \int_{n^K}^{\hat{n}}\frac{\psi'(n)}{u_c}f(n)\exp\left(\int_{n^0}^{n}\frac{u_{nc}}{u_c}d\nu\right)dn\end{aligned} \tag{C.21}$$

Because $\varphi(\hat{n}) = 0$ and because $\varphi(n^k)$ is nonpositive, it follows that

$$\int_{n^k}^{\hat{n}}\frac{\lambda - u_c}{u_c}f(n)e^{\int_{n^0}^{n}\frac{u_{nc}}{u_c}dn'}dn \geq \int_{n^k}^{\hat{n}}\frac{\psi'(n)}{u_c}f(n)e^{\int_{n^0}^{n}\frac{u_{nc}}{u_c}dn'}dn \tag{C.22}$$

for all k. Lemma C.1 shows that the right-hand side of (C.22) is equal to zero for all k. Thus,

$$\int_{n^k}^{\hat{n}}\frac{\lambda - u_c}{u_c}f(n)e^{\int_{n^0}^{n}\frac{u_{nc}}{u_c}dn'}dn \geq 0 \tag{C.23}$$

for all k. Because (C.23) holds regardless of k, it follows that, for some sequence $\{n_j\}$ converging to $\hat{n}$ from below, one has $\lambda \geq u_c(c(n_j), y(n_j), n_j)$ for all j. Again by the continuity of the allocation, (C.13) follows.

To establish (C.12), it now suffices to observe that, by Lemma 6.5 in the text, the outcome $(c(\hat{n}), y(\hat{n}))$ is efficient for $\hat{n}$. Because $\hat{c}(\hat{n}) > 0$ and $\hat{y}(\hat{n}) > 0$, this yields

$$u_c(c(\hat{n}), y(\hat{n}), \hat{n}) = |u_y(c(\hat{n}), y(\hat{n}), \hat{n})|.$$

■

Lemma C.4 *For any $n \in (n^0, n^1)$, if $c(n) > 0$, then at least one of the costate variables $\varphi(n)$, $\psi(n)$ is nonzero.*

Proof. If the lemma is false, there exists $\hat{n} \in (n^0, n^1)$ such that $c(\hat{n}) > 0$ and $\varphi(\hat{n}) = \psi(\hat{n}) = 0$. By Lemmas 6.4 and 6.5 in the text, the premises of part (b) of condition DR are satisfied for $n = \hat{n}$ and any $n' > \hat{n}$ that is sufficiently close to $\hat{n}$. For any such n', one therefore has

$$|u_y(c(\hat{n}), y(\hat{n}), \hat{n})| > \min(u_c(c(n'), y(n'), n'), \left|u_y(c(n'), y(n'), n')\right|) \text{ if } y(\hat{n}) > 0 \tag{C.24}$$

and

$$u_c(c(\hat{n}), y(\hat{n}), \hat{n}) > \min(u_c(c(n'), y(n'), n'), \left|u_y(c(n'), y(n'), n')\right|) \text{ if } y(\hat{n}) = 0. \tag{C.25}$$

By Lemma C.3, (C.24) and (C.25) imply

$$\lambda > \min(u_c(c(n'), y(n'), n'), \left|u_y(c(n'), y(n'), n')\right|) \tag{C.26}$$

for any $n' > \hat{n}$ that is sufficiently close to $\hat{n}$.

Because $c(\hat{n}) > 0$, condition (6.14) in the text must hold as an equation for $n' > \hat{n}$. Thus,

$$\psi'(n') + \varphi'(n')u_c + \varphi(n')u_{cn} = (\lambda - u_c)f(n') \tag{C.27}$$

for $n' > \hat{n}$. By condition (6.15) in the text, one also has

$$\varphi'(n')u_y + \varphi(n')u_{yn} \le -(\lambda + u_y)f(n') \tag{C.28}$$

for $n' > \hat{n}$.

I distinguish three cases and will give separate arguments for each of them.

Case 1: There exists $\bar{n} > \hat{n}$ such that

$$u_c(c(n'), y(n'), n') \le \left|u_y(c(n'), y(n'), n')\right| \tag{C.29}$$

for all $n' \in (\hat{n}, \bar{n}]$. In this case, (C.26) implies $\lambda > u_c(c(n'), y(n'), n')$, and (C.27) implies

$$\psi'(n') + \varphi'(n')u_c + \varphi(n')u_{cn} > 0 \tag{C.30}$$

for all $n' \in (\hat{n}, \bar{n}]$. Upon multiplying this inequality by $\frac{1}{u_c}\exp(\int_{\hat{n}}^{n'} \frac{u_{cn}}{u_c} d\nu)$, one can rewrite (C.30) in the form

$$\frac{\psi'(n')}{u_c}\exp(\int_{\hat{n}}^{n'} \frac{u_{cn}}{u_c} d\nu) + \frac{d}{dn'}\left(\varphi(n')\exp(\int_{\hat{n}}^{n'} \frac{u_{cn}}{u_c} d\nu)\right) > 0. \tag{C.31}$$

By integration between $\hat{n}$ and $n \in (\hat{n}, \bar{n}]$, one obtains

$$\int_{\hat{n}}^{n} \frac{\psi'(n')}{u_c} \exp(\int_{\hat{n}}^{n'} \frac{u_{cn}}{u_c} d\nu) dn' + \varphi(n) \exp(\int_{\hat{n}}^{n} \frac{u_{cn}}{u_c} d\nu) - \varphi(\hat{n}) > 0. \quad \text{(C.32)}$$

Because $\varphi(n)$ is nonpositive and $\varphi(\hat{n}) = 0$, it follows that

$$\int_{\hat{n}}^{n} \frac{\psi'(n')}{u_c} \exp(\int_{\hat{n}}^{n'} \frac{u_{cn}}{u_c} d\nu) dn' > 0$$

or

$$\int_{n}^{\hat{n}} \frac{\psi'(n')}{u_c} \exp(\int_{\hat{n}}^{n'} \frac{u_{cn}}{u_c} d\nu) dn' < 0, \quad \text{(C.33)}$$

which is incompatible with Lemma 5.2. The assumption that, for some $\hat{n} \in (n^0, n^1)$ and $\bar{n} > \hat{n}$, one has $c(\hat{n}) > 0$, $\varphi(\hat{n}) = \psi(\hat{n}) = 0$, and $u_c(c(n'), y(n'), n') \leq |u_y(c(n'), y(n'), n')|$ for all $n' \in (\hat{n}, \bar{n}]$ has thus led to a contradiction and must be false.

Case 2: There exists $\bar{n} > \hat{n}$ such that

$$u_c(c(n'), y(n'), n') \geq \left| u_y(c(n'), y(n'), n') \right| \quad \text{(C.34)}$$

for all $n' \in (\hat{n}, \bar{n}]$. In this case, (C.26) implies $\lambda > |u_y(c(n'), y(n'), n')|$, and (C.28) implies

$$\varphi'(n') u_y + \varphi(n') u_{yn} < 0 \quad \text{(C.35)}$$

for all $n' \in (\hat{n}, \bar{n}]$. Upon multiplying this inequality by $\frac{1}{u_y} \exp(\int_{\hat{n}}^{n'} \frac{u_{yn}}{u_y} d\nu) < 0$, one can rewrite (C.35) in the form

$$\frac{d}{dn'} \left(\varphi(n') \exp(\int_{\hat{n}}^{n'} \frac{u_{yn}}{u_y} d\nu) \right) > 0. \quad \text{(C.36)}$$

By integration between $\hat{n}$ and $n \in (\hat{n}, \bar{n}]$, one obtains

$$\varphi(n) \exp(\int_{\hat{n}}^{n'} \frac{u_{yn}}{u_y} d\nu) - \varphi(\hat{n}) > 0. \quad \text{(C.37)}$$

Thus, $\varphi(\hat{n}) = 0$ implies $\varphi(n) > 0$ for $n \in (\hat{n}, \bar{n}]$. This is incompatible with the first-order condition (6.8) for the slack $s(n)$ in downward incentive compatibility. The assumption that, for some $\hat{n} \in (n^0, n^1)$ and $\bar{n} > \hat{n}$, one has $c(\hat{n}) > 0$, $\varphi(\hat{n}) = \psi(\hat{n}) = 0$, and $u_c(c(n'), y(n'), n') \geq |u_y(c(n'), y(n'), n')|$ for all $n' \in (\hat{n}, \bar{n}]$ has thus also led to a contradiction and must also be false.

Case 3: There exist sequences $\{n^k\}$ and $\{n^j\}$ converging to $\hat{n}$ from above such that

$$u_c(c(n^k), y(n^k), n^k) < \left|u_y(c(n^k), y(n^k), n^k)\right| \tag{C.38}$$

for all k and

$$u_c(c(n^j), y(n^j), n^j) < \left|u_y(c(n^j), y(n^j), n^j)\right| \tag{C.39}$$

for all j. In this case, by continuity, there also exists a sequence $\{n^\ell\}$ that converges to $\hat{n}$ from above such that

$$u_c(c(n^\ell), y(n^\ell), n^\ell) = \left|u_y(c(n^\ell), y(n^\ell), n^\ell)\right| \tag{C.40}$$

for all ℓ. By continuity, it follows that

$$u_c(c(\hat{n}), y(\hat{n}), \hat{n}) = \left|u_y(c(\hat{n}), y(\hat{n}), \hat{n})\right|. \tag{C.41}$$

For any ℓ, one also has

$$u_c(c(n^\ell), y(n^\ell), n^\ell) = u_c(c(\hat{n}), y(\hat{n}), \hat{n}) + \int_{\hat{n}}^{n^\ell} \left[u_{cc}q + u_{cy}z + u_{cn}\right] dn' \tag{C.42}$$

and

$$u_y(c(n^\ell), y(n^\ell), n^\ell) = u_y(c(\hat{n}), y(\hat{n}), \hat{n}) + \int_{\hat{n}}^{n^\ell} \left[u_{yc}q + u_{yy}z + u_{yn}\right] dn', \tag{C.43}$$

where $q(\cdot)$ and $z(\cdot)$ are the derivatives of $c(\cdot)$ and $y(\cdot)$. Thus, (C.40) and (C.41) yield

$$\int_{\hat{n}}^{n^\ell} \left[u_{cc}q + u_{cy}z + u_{cn} + u_{yc}q + u_{yy}z + u_{yn}\right] dn' = 0. \tag{C.44}$$

By SSCC, (C.41) implies

$$u_{cn}(c(\hat{n}), y(\hat{n}), \hat{n}) + u_{yn}(c(\hat{n}), y(\hat{n}), \hat{n}) > 0. \tag{C.45}$$

By continuity, it follows that, for some $\varepsilon > 0$, one has

$$u_{cn}(c(n'), y(n'), n') + u_{yn}(c(n'), y(n'), n') > \varepsilon \tag{C.46}$$

for all n' in a sufficiently small neighbourhood of $\hat{n}$. For any sufficiently large ℓ, one therefore has

$$\int_{\hat{n}}^{n^\ell} [u_{cn} + u_{yn}]\, dn' \geq \varepsilon(n^\ell - \hat{n}). \tag{C.47}$$

Upon combining this inequality with (C.44), one obtains

$$\int_{\hat{n}}^{n^\ell} [u_{cc}q + u_{cy}z + u_{yc}q + u_{yy}z]\, dn' \leq -\varepsilon(n^\ell - \hat{n}). \tag{C.48}$$

If one divides this inequality by $(n^\ell - \hat{n})$ and takes limits as ℓ becomes large and n^ℓ converges to $\hat{n}$, one finds that

$$[u_{cc} + u_{yc}]q(\hat{n}+) + [u_{cy} + u_{yy}]z(\hat{n}+) \leq -\varepsilon, \tag{C.49}$$

where $q(\hat{n}+)$ and $z(\hat{n}+)$ are the limits of $q(n')$ and $z(n')$ as n' converges to $\hat{n}$ from above. By Lemma 6.3, $q(\hat{n}+) = 0$ would imply $z(\hat{n}+) = 0$. Therefore, (C.49) yields $q(\hat{n}+) > 0$ and, indeed, $q(n') > 0$ for any $n' > \hat{n}$ that is sufficiently close to $\hat{n}$.

It follows that, for any $n' > \hat{n}$ that is sufficiently close to $\hat{n}$, one has $\psi(n') = 0$ and $\psi'(n') = 0$. For such n', (C.27) becomes

$$\varphi'(n')u_c + \varphi(n')u_{cn} = (\lambda - u_c)f(n'), \tag{C.50}$$

which, by SSCC and the nonpositivity of $\varphi(n')$, implies

$$\varphi'(n') + \varphi(n')\frac{u_{yn}}{u_y} \geq \frac{\lambda - u_c}{u_c}f(n'). \tag{C.51}$$

In combination with (C.28), this implies

$$\begin{aligned}\varphi'(n') + \varphi(n')\frac{u_{yn}}{u_y} \quad &\geq \quad \max\left(\frac{\lambda - u_c}{u_c}f(n'), \frac{\lambda - |u_y|}{|u_y|}f(n')\right) \\ &= \quad \frac{\lambda - \min(u_c, |u_y|)}{\min(u_c, |u_y|)}f(n'). \end{aligned} \tag{C.52}$$

By (C.26), (C.52) implies

$$\varphi'(n') + \varphi(n')\frac{u_{yn}}{u_y} > 0. \tag{C.53}$$

As discussed in the text, multiplication by $\exp(\int_{\hat{n}}^{n'} \frac{u_{yn}}{u_y} d\nu)$ and integration yield

$$\varphi(n)\exp(\int_{\hat{n}}^{n} \frac{u_{yn}}{u_y} d\nu) - \varphi(\hat{n}) > 0, \tag{C.54}$$

so, with $\varphi(\hat{n}) = 0$, one must have $\varphi(n) > 0$ for any $n > \hat{n}$ that is sufficiently close to $\hat{n}$. This is incompatible with the first-order condition (6.8) for the slack $s(n)$ in downward incentive compatibility. The assumption that, for some $\hat{n} \in (n^0, n^1)$ and $\bar{n} > \hat{n}$, one has $c(\hat{n}) > 0$, $\varphi(\hat{n}) = \psi(\hat{n}) = 0$, as well as

$$u_c(c(n^k), y(n^k), n^k) < \left| u_y(c(n^k), y(n^k), n^k) \right| \tag{C.55}$$

and

$$u_c(c(n^j), y(n^j), n^j) < \left| u_y(c(n^j), y(n^j), n^j) \right| \tag{C.56}$$

for all k and j, for sequences $\{n^k\}$ and $\{n^j\}$ converging to $\hat{n}$ from above, has thus also led to a contradiction and must be false.

Because Cases 1 - 3 cover all possibilities, the assumption that, for some $\hat{n} \in (n^0, n^1)$ and $\bar{n} > \hat{n}$, one has $c(\hat{n}) > 0$, $\varphi(\hat{n}) = \psi(\hat{n}) = 0$ must be false. $\blacksquare$

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