Preprints of the
Max Planck Institute
for Research on Collective Goods
Bonn
2004/6

Risk Aversion in the Small and in the Large When Outcomes are Multidimensional

Martin F. Hellwig

Preprints of the
Max Planck Institute
for Research on Collective Goods Bonn 2004/6

Risk Aversion in the Small and in the Large When Outcomes are Multidimensional

Martin F. Hellwig

June 2004

Max Planck Institute for Research on Collective Goods, Kurt-Schumacher-Str. 10, D-53113 Bonn
http://www.mpp-rdg.mpg.de

Risk Aversion in the Small and in the Large When Outcomes Are Multidimensional*

Martin F. Hellwig
Max Planck Institute for Research on Collective Goods
Kurt-Schumacher-Str. 10, D - 53113 Bonn, Germany
and
Sonderforschungsbereich 504, University of Mannheim
D - 68131 Mannheim, Germany
hellwig@mpp-rdg.mpg.de

June 15, 2004

Abstract

The paper discusses criteria for comparing risk aversion of decision makers when outcomes are multidimensional. A weak concept, "commodity specific greater risk aversion", is based on the comparison of risk premia paid in a specified commodity. A stronger concept, "uniformly greater risk aversion" is based on the comparison of risk premia regardless of what commodities are used for payment. Neither concept presumes that von Neumann-Morgenstern utility functions are ordinally equivalent. Nonincreasing consumption specific risk aversion is shown to be sufficient to make randomization undesirable in an agency problem with hidden characteristics.

Key Words: Multidimensional Risks, Risk Aversion, Risk Premia, Randomization in Incentive Schemes.

JEL Classification: D81, D82

1 Introduction

The concept of risk aversion is one of the most important concepts in the theory of decision making under uncertainty. Measures of risk aversion have

*Research support from Deutsche Forschungsgemeinschaft through Sonderforschungsbereich 504 is gratefully acknowledged.

been developed by Arrow (1965) and Pratt (1964) for choices involving unidimensional outcomes. The measures permit a certain quantification of the concept, providing a basis for the comparative analysis of the risk choices of different people as well as the comparative-static analysis of the dependence of risk choices on parameters such as wealth.

The present paper extends the Arrow-Pratt concept of absolute risk aversion to choices involving multidimensional outcomes. It also provides an application of the extended risk aversion measure in the analysis of an incentive problem with hidden characteristics.

Measurement of risk aversion with respect to multidimensional outcomes has previously been studied by Kihlstrom and Mirman (1974, 1981). To define the notion that one von Neumann-Morgenstern utility function on an n-dimensional outcome space involves greater risk aversion than another, they assumed that both utility functions induce the *same* preference ordering over outcomes, i.e. that utility functions differ only in the cardinal representations of a given preference ordering. The approach developed in this paper involves no such assumption. However, in those instances where the Kihlstrom-Mirman assumption is satisfied, the notion of risk aversion that is proposed here coincides with theirs.

A concept of "greater risk aversion" that does not presume identical preferences over the underlying outcome space is useful in the analysis of incentive problems with hidden characteristics. In situations involving hidden characteristics incentive devices inducing self-selection are useful precisely because people with different characteristics have different preference orderings over the given multidimensional outcome space. In these situations, the Kihlstrom-Mirman approach is *a priori* inapplicable as a tool for comparing the risk attitudes of different types. However, such a tool is needed if one is to have a clear view of the notion that, because of differences in risk attitudes, it may be desirable to use *randomized* incentive mechanisms.[1] The measure proposed here serves precisely that purpose.

To some extent, the approach developed here confirms the view of Kihlstrom and Mirman that differences in induced orderings over outcomes cause difficulties for the comparative assessment of risk aversion. However, the difficulties are less serious than they suggested.

Going back to the approach of Arrow and Pratt, consider the assessment of risk aversion in terms of risk premia, saying that one agent is more risk

[1] Maskin and Riley (1984), Stiglitz (1982), Fudenberg and Tirole (1991). For settings involving unidimensional outcomes, Maskin and Riley (1984) as well as Matthews and Moore (1987) show that nonincreasing absolute risk aversion makes randomization undesirable.

averse than another if the amount that he is willing to pay in order to be spared the randomness in a given lottery exceeds the amount that the other agent is willing to pay for the same purpose. In a multidimensional context, this approach raises the question of what it means to pay a certain amount in order to be spared the randomness in a given lottery. Is the risk premium to be paid in apples or in pears? If the von Neumann-Morgenstern utility functions of the two agents in question induce different preference orderings over the underlying outcome space, the means of payment can make a difference. One agent is deemed to be more risk averse than another if the risk premium has to be paid in apples and to be less risk averse if the risk premium has to be paid in pears. The Kihlstrom-Mirman assumption of identical preference orderings over the underlying outcome space serves to exclude this paradox.

The approach developed here is based on the view that it is not necessary to exclude the paradox in order to talk sensibly about comparisons of risk aversion in a multidimensional context. First, if one is explicit about the units in which the risk premium is to be paid, one obtains a concept of "premium-specific greater risk aversion" which does not give rise to a paradox at all - and which turns out to be useful in studying incentive mechanisms. Second, if one considers the notion of premium-specific greater risk aversion to be too weak, one can define a concept of "uniformly greater risk aversion", which applies if one has premium-specific greater risk aversion regardless of the specification of the units in which the premium is to be paid. Under the Kihlstrom-Mirman assumption of identical preference orderings over the underlying outcome space, their concept of "greater risk aversion" coincides with the concept of "uniformly greater risk aversion" proposed here. However, the later concept has meaning even if the Kihlstrom-Mirman assumption is not satisfied.

In the following, Section 2 introduces the concept of "premium-specific greater risk aversion" and provides an analogue of Pratt's characterization result for the unidimensional case. Section 3 indicates how the concept can be useful in analysing incentive problems with hidden characteristics. Section 4 discusses the role of premium specificity versus uniformity in risk aversion. This section provides (i) an example where the comparison of risk attitudes depends on the specification of the risk premium, (ii) a definition and characterization result for the concept of "uniformly greater risk aversion", and (iii) an example of utility functions that are comparable by the criterion of "uniformly greater risk aversion" even though they do not satisfy the Kihlstrom-Mirman assumption of ordinal equivalence.

2 Commodity Specific Risk Premia and Risk Aversion

Consider a decision maker choosing between lotteries with outcome realizations $\mathbf{x} \in \Re^n$. Assume that the decision maker treats any one lottery as a random variable $\tilde{x}$ on some underlying probability space $(\Omega, \mathcal{F}, \nu)$ and that he assesses lotteries according to the expected-utility functional

$$Eu(\tilde{\mathbf{x}}) = \int_\Omega u(\mathbf{x}(\omega))d\nu(\omega), \tag{2.1}$$

where $u : \Re^n \to \Re$ is a strictly increasing, concave and twice continuously differentiable function with first derivatives $u_1, ..., u_n$ and Hessian $D^2u = (u_{ij})$.

For convenience, I refer to the different dimensions of the outcome vectors as "commodities". For $i = 1, ..., n$, the *i-risk-premium* indicates the number $\pi_i(\tilde{\mathbf{x}}, u)$ of units of commodity i that the decision maker would be willing to give up in order to avoid the randomness inherent in the lottery $\tilde{\mathbf{x}}$. Formally, $\pi_i(\tilde{\mathbf{x}}, u)$ is defined to be the solution to the equation

$$Eu(\tilde{\mathbf{x}}) = u(E\tilde{x}_1, ...E\tilde{x}_i - \pi_i, ..., E\tilde{x}_n). \tag{2.2}$$

A utility function u is said to exhibit *i-premium specific weakly greater risk aversion* than an alternative utility function v, if and only if one has

$$\pi_i(\tilde{\mathbf{x}}, u) \geq \pi_i(\tilde{\mathbf{x}}, v) \tag{2.3}$$

for every lottery $\tilde{\mathbf{x}}$. The utility function u is said to exhibit *i-premium specific strictly greater risk aversion* than v if for every $\tilde{\mathbf{x}}$ the inequality in (2.3) is strict. The following result provides a multidimensional analogue of Pratt's theorem.

Theorem 2.1 *For any i and any two von Neumann-Morgenstern utility functions u and v, the following statements are equivalent:*

(a) u exhibits i-premium specific weakly greater risk aversion than v;

(b) there exists a concave function φ^i such that for all $\mathbf{x} \in \Re$,

$$u(\mathbf{x}) = \varphi^i(x_1, ..., x_{i-1}, v(x_1, ..., x_n), x_{i+1}, ..., x_n); \tag{2.4}$$

(c) the matrix

$$B^i(\mathbf{x}|u, v) = -\left[\frac{1}{u_i(\mathbf{x})}D^2u(\mathbf{x}) - \frac{1}{v_i(\mathbf{x})}D^2v(\mathbf{x})\right] \tag{2.5}$$

is positive semidefinite for every $\mathbf{x} \in \Re$.

Following Pratt's line of argument, I will prove the implications $(b) \Longrightarrow (a) \Longrightarrow (c) \Longrightarrow (b)$. For clarity of exposition, each implication is stated as a separate lemma. For the implications $(b) \Longrightarrow (a)$ and $(c) \Longrightarrow (b)$, the lemmas also give the corresponding "strict" versions. For the implication $(a) \Longrightarrow (c)$, I have not found a "strict" version that is simple to state, let alone sufficient to establish an equivalence like the one obtained for the weak version.

Lemma 2.2 $(b) \Longrightarrow (a)$. *Moreover, if φ^i is strictly concave, then u exhibits i-premium specific strictly greater risk aversion than v.*[2]

Proof. For any nondegenerate lottery $\tilde{\mathbf{x}}$, (2.4) implies

$$Eu(\tilde{\mathbf{x}}) = E\varphi_i(\tilde{x}_1, ..., \tilde{x}_{i-1}, v(\tilde{x}_1, ..., \tilde{x}_n), \tilde{x}_{i+1}, ..., \tilde{x}_n). \tag{2.6}$$

By the concavity of φ_i, it follows that

$$Eu(\tilde{\mathbf{x}}) \leq \varphi_i(E\tilde{x}_1, ..., E\tilde{x}_{i-1}, Ev(\tilde{x}_1, ..., \tilde{x}_n), E\tilde{x}_{i+1}, ..., E\tilde{x}_n). \tag{2.7}$$

By the definition of $\pi_i(\tilde{\mathbf{x}}, v)$, (2.7) is equivalent to the inequality

$$Eu(\tilde{\mathbf{x}}) \leq \varphi_i(E\tilde{x}_1, ..., E\tilde{x}_{i-1}, v(E\tilde{x}_1, ..., E\tilde{x}_i - \pi_i(\tilde{\mathbf{x}}, v), E\tilde{x}_n), E\tilde{x}_{i+1}, ..., E\tilde{x}_n),$$

so another application of (2.4) yields

$$Eu(\tilde{\mathbf{x}}) \leq u(E\tilde{x}_1, ..., E\tilde{x}_{i-1}, E\tilde{x}_i - \pi_i(\tilde{\mathbf{x}}, v), E\tilde{x}_{i+1}, ..., E\tilde{x}_n). \tag{2.8}$$

By the definition of $\pi_i(\tilde{\mathbf{x}}, u)$ and the monotonicity of u, it follows that $\pi_i(\tilde{\mathbf{x}}, u) \geq \pi_i(\tilde{\mathbf{x}}, v)$. Thus, u exhibits *i-premium specific weakly greater risk aversion* than v. If φ^i is strictly concave, the inequality in (2.7) and with it all subsequent inequalities are strict. ■

Lemma 2.3 $(a) \Longrightarrow (c)$.

Proof. Fix $\mathbf{x} \in \Re$ and, for any $h > 0$ and any random variable $\tilde{\mathbf{y}}$ with $E\tilde{\mathbf{y}} = \mathbf{0}$, consider the lottery $\tilde{\mathbf{x}}(h, \tilde{\mathbf{y}}) = \mathbf{x} + h\tilde{\mathbf{y}}$. Given that $Ey_j = 0$ for all j, the *i-risk-premium* $\pi_i(\tilde{\mathbf{x}}(h, \tilde{\mathbf{y}}), u)$ is given by the equation

$$Eu(\mathbf{x} + h\tilde{\mathbf{y}}) = u(x_1, ..., x_i - \pi_i(\tilde{\mathbf{x}}(h, \tilde{\mathbf{y}}), u), ..., x_n). \tag{2.9}$$

[2] For lotteries taking the form $\tilde{x} = \tilde{\lambda}x_1 + (1 - \tilde{\lambda})x_2$ and u, v satisfying $u = \varphi \circ v$, with φ strictly concave, this is the result of Kihlstrom and Mirman (1974).

The left-hand side of (2.9) can be written as

$$Eu(\mathbf{x} + h\tilde{\mathbf{y}}) = u(\mathbf{x}) + \frac{h^2}{2} E\left[\tilde{\mathbf{y}}' D^2 u(\mathbf{x})\tilde{\mathbf{y}}\right] + o(h^2), \tag{2.10}$$

the right-hand side as

$$\begin{aligned} & u(x_1, ..., x_i - \pi_i(\tilde{\mathbf{x}}(h,\tilde{\mathbf{y}}), u), ..., x_n) \\ = \;& u(\mathbf{x}) - \pi_i(\tilde{\mathbf{x}}(h,\tilde{\mathbf{y}}), u)\; u_i(\mathbf{x}) + o(\pi_i(\tilde{\mathbf{x}}(h,\tilde{\mathbf{y}}), u)), \end{aligned} \tag{2.11}$$

where $o(h^2)$ and $o(\pi_i(\tilde{\mathbf{x}}(h,\tilde{\mathbf{y}}), u))$ are terms going to zero faster than h^2 and $\pi_i(\tilde{\mathbf{x}}(h,\tilde{\mathbf{y}}), u)$. By standard arguments, as in Pratt (1964), (2.9) - (2.11) imply $\lim_{h\to 0} \pi_i(\tilde{\mathbf{x}}(h,\tilde{\mathbf{y}}), u) = 0$, $\lim_{h\to 0} \pi_i(\tilde{\mathbf{x}}(h,\tilde{\mathbf{y}}), u)/h = 0$, and

$$\lim_{h\to 0} \frac{1}{h^2} \pi_i(\tilde{\mathbf{x}}(h,\tilde{\mathbf{y}}), u) = -\frac{1}{2\, u_i(\mathbf{x})} E\left[\tilde{\mathbf{y}}' D^2 u(\mathbf{x})\tilde{\mathbf{y}}\right]. \tag{2.12}$$

By the same reasoning, one also has

$$\lim_{h\to 0} \frac{1}{h^2} \pi_i(\tilde{\mathbf{x}}(h,\tilde{\mathbf{y}}), v) = -\frac{1}{2\, v_i(\mathbf{x})} E\left[\tilde{\mathbf{y}}' D^2 v(\mathbf{x})\tilde{\mathbf{y}}\right],$$

so $\pi_i(\tilde{\mathbf{x}}(h,\tilde{\mathbf{y}}), u) > \pi_i(\tilde{\mathbf{x}}(h,\tilde{\mathbf{y}}), v)$ for all h and $\tilde{\mathbf{y}}$ implies

$$-\frac{1}{u_i(\mathbf{x})} E\left[\tilde{\mathbf{y}}' D^2 u(\mathbf{x})\tilde{\mathbf{y}}\right] \geq -\frac{1}{v_i(\mathbf{x})} E\left[\tilde{\mathbf{y}}' D^2 v(\mathbf{x})\tilde{\mathbf{y}}\right] \tag{2.13}$$

for all $\tilde{\mathbf{y}}$. If $\tilde{\mathbf{y}}$ is such that for some $\mathbf{z} \in \Re^n$, $\tilde{\mathbf{y}} = \mathbf{z}$ with probability $\frac{1}{2}$ and $\tilde{\mathbf{y}} = -\mathbf{z}$ with probability $\frac{1}{2}$, (2.13) becomes

$$-\mathbf{z}' \frac{1}{u_i(\mathbf{x})} D^2 u(\mathbf{x})\; \mathbf{z} \geq -\,\mathbf{z}' \frac{1}{v_i(\mathbf{x})} D^2 v(\mathbf{x})\; \mathbf{z}. \tag{2.14}$$

If (2.14) is to hold for all $\mathbf{z} \in \Re^n$, the matrix (2.5) must be positive semi-definite. ■

Lemma 2.4 *(c)* $\Longrightarrow$ *(b). Moreover, if for every* $\mathbf{x}_0$ *and* $\mathbf{x}_0$ *in* $\Re^n$*, the matrix* $B^i(\mathbf{x}_\lambda|u,v)$ *that is associated with the convex combination* $\mathbf{x}_\lambda = \lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_0$ *is positive definite for a nonnull set of* $\lambda \in [0,1]$*, then* φ^i *is strictly concave.*

Proof. For any $\mathbf{x} \in \Re^n$ and $v = v(\mathbf{x})$, define

$$\varphi^i(x_1, ..., x_{i-1}, v, x_{i+1} ..., x_n) = u(\mathbf{x}). \tag{2.15}$$

Then φ^i is well defined on the set of vectors $(x_1, ..., x_{i-1}, v, x_{i+1} ..., x_n)$ such that $v = v(x_1, ..., x_{i-1}, x_i, x_{i+1} ..., x_n)$ for some x_i. Moreover, φ^i is twice continuously differentiable. Its first derivatives are computed as

$$\varphi^i_i = \frac{u_i}{v_i}, \tag{2.16}$$

$$\varphi^i_j = u_j - \varphi^i_i v_j \tag{2.17}$$

for $j \neq i$. The second derivatives of φ^i are found to satisfy

$$\varphi^i_{ii} = \frac{1}{v_i^2}[u_{ii} - \frac{u_i}{v_i} v_{ii}] = \frac{u_i}{v_i^2}\left[\frac{u_{ii}}{u_i} - \frac{v_{ii}}{v_i}\right], \tag{2.18}$$

$$\varphi^i_{ij} = \frac{1}{v_i}[u_{ij} - \frac{u_i}{v_i} v_{ij} - \varphi^i_{ii} v_i v_j] = \frac{u_i}{v_i}\left[\frac{u_{ij}}{u_i} - \frac{v_{ij}}{v_i}\right] - \varphi^i_{ii} v_j \tag{2.19}$$

for $j \neq i$, and

$$\varphi^i_{jk} = u_{jk} - \frac{u_i}{v_i} v_{jk} - \varphi^i_{ik} v_j - \varphi^i_{ij} v_k - \varphi^i_{ii} v_j v_k \tag{2.20}$$

for $j \neq i$ and $k \neq i$. Upon writing

$$\beta^i_{jk} := -\left[\frac{u_{jk}}{u_i} - \frac{v_{jk}}{v_i}\right] \tag{2.21}$$

for the typical element of the matrix $B^i(\mathbf{x}|u, v)$, one finds that (2.18) - (2.20) can be rewritten as

$$\varphi^i_{ii} = -\frac{u_i}{v_i^2}\beta^i_{ii}, \tag{2.22}$$

$$\varphi^i_{ij} = -\frac{u_i}{v_i}\beta^i_{ij} + \frac{u_i v_j}{v_i^2}\beta^i_{ii} \tag{2.23}$$

for $j \neq i$, and

$$\varphi^i_{jk} = -u^i_i \beta^i_{jk} + \frac{u_i}{v_i}\beta^i_{ij} v_k + \frac{u_i}{v_i}\beta^i_{ik} v_j + \frac{u_i}{v_i^2}\beta^i_{ii} v_j v_k \tag{2.24}$$

for $j \neq i$ and $k \neq i$. For any vector $\mathbf{z} \in \Re^n$, one therefore computes

$$\begin{aligned}\sum_{j=1}^{n}\sum_{k=1}^{n} z_j \varphi^i_{jk} z_k \quad = \quad & -\frac{u_i}{v_i^2}\beta^i_{ii}\left[z_i^2 - z_i \sum_{j\neq i} z_j v_j - z_i \sum_{k \neq i} z_k v_k + \sum_{j\neq i}\sum_{k\neq i} z_j v_j z_k v_k\right] \\ & -\frac{u_i}{v_i}\sum_{j\neq i} z_j \beta^i_{ij}(z_i - \sum_{k\neq i} z_k v_k) - \frac{u_i}{v_i}(z_i - \sum_{j\neq i} z_j v_j)\sum_{k\neq i} z_k \beta^i_{ik} \\ & -u_i \sum_{j\neq i}\sum_{k\neq i} z_j \beta^i_{jk} z_k,\end{aligned}$$

or

$$\sum_{j=1}^{n}\sum_{k=1}^{n} z_j \varphi^i_{jk} z_k = -u_i \sum_{j=1}^{n}\sum_{k=1}^{n} \hat{z}_j(\mathbf{z})\beta^i_{jk}\hat{z}_k(\mathbf{z}), \tag{2.25}$$

where $\hat{z}_i(\mathbf{z}) = (z_i - \sum_{k\neq i} z_k v_k)$ and, for $j \neq i$, $\hat{z}_j(\mathbf{z}) = z_j$. If statement (c) holds, the quadratic form $\sum_{j=1}^{n}\sum_{k=1}^{n} z_j \varphi^i_{jk} z_k$ is nonpositive whenever $\mathbf{z} \neq \mathbf{0}$. It follows that φ^i is concave.

To obtain strict concavity, observe that, for any $\mathbf{x}_0$ and $\mathbf{x}_1 = \mathbf{x}_0 + \mathbf{y}$ in $\Re^n$, one has

$$\varphi^i(\mathbf{x}_1) - \varphi^i(\mathbf{x}_0) = \int_0^1 \sum_{j=1}^{n} \varphi^i_j(\mathbf{x}_0 + \alpha\mathbf{y}) y_j \, d\alpha$$

$$= \sum_{j=1}^{n} \varphi^i_j(\mathbf{x}_0) y_j + \int_0^1 \int_0^{\alpha} \sum_{j=1}^{n} y_j \varphi^i_{jk}(\mathbf{x}_0 + \lambda\mathbf{y}) y_k \, d\lambda d\alpha. \tag{2.26}$$

If $\mathbf{x}_1 \neq \mathbf{x}_0$ and if the matrix $B^i(\mathbf{x}_\lambda|u, v)$ that is associated with the convex combination $\mathbf{x}_\lambda = \lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_0 = \mathbf{x}_0 + \lambda\mathbf{y}$ is positive definite for a nonnull set of $\lambda \in [0, 1]$, (2.26) and (2.25) imply

$$\varphi^i(\mathbf{x}_1) - \varphi^i(\mathbf{x}_0) < \sum_{j=1}^{n} \varphi^i_j(\mathbf{x}_0) y_j = \sum_{j=1}^{n} \varphi^i_j(\mathbf{x}_0)(x_{1j} - x_{0j}). \tag{2.27}$$

Thus, in this case, φ^i is strictly concave. ■

Theorem 2.1 suggests that the matrix

$$A^i(\mathbf{x}|u) := -\frac{1}{u_i(\mathbf{x})} D^2 u(\mathbf{x}) \tag{2.28}$$

be regarded as a quantitative local measure of i-premium specific risk aversion, much like the measure $-u''/u'$ of relative curvature measures absolute risk aversion in the unidimensional case. The suggestion is supported by the observation that, starting from $\mathbf{x}$, by (2.12), for any mean-zero random vector $\tilde{\mathbf{y}}$, the i-risk-premium associated with small lotteries in the direction of $\tilde{\mathbf{y}}$ satisfies

$$\lim_{h\to 0}\frac{1}{h^2}\pi_i(\tilde{\mathbf{x}}(h,\tilde{\mathbf{y}}),u) = -\frac{1}{2}E\left[\tilde{\mathbf{y}}' A^i(\mathbf{x}|u)\tilde{\mathbf{y}}\right] = -\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\sigma_{jk}\frac{u_{jk}}{u_i}, \qquad (2.29)$$

where $\sigma_{jk} = E\tilde{y}_j\tilde{y}_k$ for $j = 1, ..., n$ and $k = 1, ..., n$. Locally, starting from $\mathbf{x}$, the i-risk-premium associated with small lotteries in the direction of $\tilde{\mathbf{y}}$ depends only on the size of the lottery, the variance-covariance matrix of $\tilde{\mathbf{y}}$, and the risk aversion matrix $A^i(\mathbf{x}|u)$. Moreover the i-risk-premium increases in the risk aversion matrix in the sense that if quadratic forms involving $A^i(\mathbf{x}|u)$ exceed the corresponding quadratic forms involving $A^i(\mathbf{x}|v)$, then i-risk-premia for a person with utility function u are greater than i-risk-premia for a person with utility function v.

3 Decreasing Premium Specific Risk Aversion and the Undesirability of Randomized Incentive Schemes

This section applies the concept of i-premium specific risk aversion to the analysis of a principal-agent problem with hidden characteristics. Let $n = 2$ and interpret good 1 as consumption and good 2 as leisure of the agent. A given pair (x_1, x_2) provides the agent with the utility $u(x_1, x_2)$, where, as before, $u : \Re^2 \to \Re$ is a strictly increasing, concave and twice continuously differentiable function with first derivatives u_1, u_2 and Hessian $D^2u = (u_{ij})$.

A principal wants the agent to produce some output y in return for a wage payment w. The principal has all the bargaining power in that he can make an ultimatum offer which the agent can only accept or reject. However, he does not have complete information: The agent is characterized by a productivity parameter θ with the interpretation that the production of the output y requires him to work $\frac{y}{\theta}$ hours, reducing his leisure to $1 - \frac{y}{\theta}$. The principal cannot observe θ or the actual working time $\frac{y}{\theta}$ of the agent. He only knows that θ is the realization of a random variable $\tilde{\theta}$ with possible values $\theta_1, ..., \theta_m$, with prior probabilities $p_1, ..., p_m$. Without loss of generality, I assume that $\theta_1 < \theta_2 < ... < \theta_m$. The agent knows his own productivity parameter. He also knows his payoff $\bar{u}$ when he does not work for the principal; this payoff is taken to be independent of θ.

Given his lack of information, the principal offers a menu of m contracts, leaving the agent to choose whichever he likes best according to his type. A *deterministic-contract menu* is simply a list $\{(w_t, y_t)\}_{t=1}^m$ of wage/output combinations such that (w_t, y_t) is the wage/output combination intended for the agent when his productivity parameter is θ_t. A *stochastic-contract menu* is a list $\{(\tilde{w}_t, \tilde{y}_t)\}_{t=1}^m$of random wage/output combinations such that $(\tilde{w}_t, \tilde{y}_t)$ is the combination intended for the agent when his productivity parameter is θ_t. The idea is that after the agent has chosen a pair $(\tilde{w}_t, \tilde{y}_t)$, the randomness in this pair is resolved and the realization (w, y) of the random pair $(\tilde{w}_t, \tilde{y}_t)$ that is observed determines the actual wage that the agent will receive and the actual output that he must provide.

Allowing for the possibility of offering stochastic-contract menus, the principal's problem is to choose the list $\{(\tilde{w}_t, \tilde{y}_t)\}_{t=1}^m$ so as to maximize his own expected profit

$$\sum_{i=1}^{m} p_t E(\tilde{y}_t - \tilde{w}_t) \tag{3.1}$$

subject to the incentive compatibility conditions that

$$Eu(\tilde{w}_t, 1 - \frac{\tilde{y}_t}{\theta_t}) \geq Eu(\tilde{w}_{t'}, 1 - \frac{\tilde{y}_{t'}}{\theta_t}) \tag{3.2}$$

for all t and all t' and subject to the individual-rationality conditions that

$$Eu(\tilde{w}_t, 1 - \frac{\tilde{y}_t}{\theta_t}) \geq \bar{u} \tag{3.3}$$

for all t.

In addition to the monotonicity, curvature, and regularity conditions that have already been mentioned, I assume that the utility function u satisfies the following assumptions:

(A1) **Strict Single-Crossing Property:** For all θ and all (w, y) and (w', y') in $\Re_+^2$ such that $y' > y$, $u(w, 1 - \frac{y}{\theta}) = u(w', 1 - \frac{y'}{\theta})$ implies $u(w, 1 - \frac{y}{\hat{\theta}}) < u(w', 1 - \frac{y'}{\hat{\theta}})$ for all $\hat{\theta} > \theta$.

(A2) **Weakly Decreasing Consumption Specific Risk Aversion:** For any θ and $\theta' > \theta$, the function $(w, y) \to u(w, 1 - \frac{y}{\theta})$ exhibits 1-premium specific weakly greater risk aversion than the function $(w, y) \to u(w, 1 - \frac{y}{\theta'})$.

Assumption (A1) is standard for problems of this type with hidden characteristics. As discussed by Milgrom and Shannon (1994) and Edlin and Shannon (1998), it is slightly weaker than the strict Spence-Mirrlees condition that the marginal rate of substitution between consumption and output provision, $\frac{u_2/\theta}{u_1}$, be a *decreasing* function of θ. It implies, in particular, that different values of the productivity parameter induce different preference orderings over outcomes in (w, y)-space.

Given this difference in preference orderings, an assessment of differences in risk aversion at different values of the productivity parameter cannot rely on the approach of Kihlstrom and Mirman (1974) which presumes identical orderings. As an alternative, Assumption (A2) uses the concept of i-premium specific risk aversion of the previous section, postulating that higher values of the productivity parameter go along with lower consumption specific risk aversion.

Because the ratio $\frac{u_2/\theta}{u_1}$ is a decreasing function of θ, Assumption (A2) is actually *weaker* than the corresponding assumption for leisure specific (2-specific) risk aversion. By Theorem 2.1, (A2) is equivalent to the assumption that the matrix $\begin{pmatrix} \frac{u_{11}}{u_1} & -\frac{u_{12}}{u_1\theta} \\ -\frac{u_{21}}{u_1\theta} & \frac{u_{22}}{u_1\theta^2} \end{pmatrix}$ be increasing in θ in the sense of statement (c) in Theorem 2.1. The corresponding requirement for leisure specific risk aversion would be that the matrix $\begin{pmatrix} \frac{u_{11}\theta}{u_2} & -\frac{u_{12}}{u_2} \\ -\frac{u_{21}}{u_2} & \frac{u_{22}}{u_2\theta} \end{pmatrix}$ be increasing in θ. Under (A1), this latter requirement is more restrictive than (A2).

Theorem 3.1 *Assume (A1) and (A2) and let u be strictly concave. Then any solution to the principal's problem involves a deterministic-contract menu, i.e., it is not optimal for the principal to offer nondegenerate randomized contracts.*

Theorem 3.1 provides a two-dimensional analogue of unidimensional results in Maskin and Riley (1984) as well as Matthews and Moore (1987). To prove Theorem 3.1, I follow the strategy of Matthews and Moore (1987) and consider the *relaxed problem* which is obtained if incentive compatibility is weakened to the *downward* incentive compatibility requirement that (3.2) has to hold for all t and $t' < t$. In this relaxed problem, the principal chooses a stochastic-contract menu $\{(\tilde{w}_t, \tilde{y}_t)\}_{t=1}^m$ so as to maximize (3.1) subject to downward incentive compatibility and individual rationality. The following lemma is straightforward.

Lemma 3.2 *Assume (A1) and (A2) and let u be strictly concave. Then any solution to the principal's relaxed problem involves a deterministic-contract menu.*

Proof. Let $\{(\tilde{w}_t, \tilde{y}_t)\}_{t=1}^m$ be any solution to the principal's relaxed problem. Consider the deterministic-contract menu $\{(\bar{w}_t, \bar{y}_t)\}_{t=1}^m$ such that for any t, $\bar{y}_t = E\tilde{y}_t$, and

$$u(\bar{w}_t, 1 - \frac{\bar{y}_t}{\theta_t}) = Eu(\tilde{w}_t, 1 - \frac{\tilde{y}_t}{\theta_t}). \tag{3.4}$$

Then trivially, for any t, the validity of (3.3) implies that $u(\bar{w}_t, 1 - \frac{\bar{y}_t}{\theta_t}) \geq \bar{u}$. The menu $\{(\bar{w}_t, \bar{y}_t)\}_{t=1}^m$ therefore is individually rational. By Assumption (A2) and (3.4), one also has

$$u(\bar{w}_t, 1 - \frac{\bar{y}_t}{\theta_{\hat{t}}}) \leq Eu(\tilde{w}_t, 1 - \frac{\tilde{y}_t}{\theta_{\hat{t}}}) \tag{3.5}$$

for all t and all $\hat{t} > t$. For any t and any $t' < t$, the validity of (3.2) and (3.4) therefore imply

$$u(\bar{w}_t, 1 - \frac{\bar{y}_t}{\theta_t}) = Eu(\tilde{w}_t, 1 - \frac{\tilde{y}_t}{\theta_t}) \geq Eu(\tilde{w}_{t'}, 1 - \frac{\tilde{y}_{t'}}{\theta_t}) \geq u(\bar{w}_{t'}, 1 - \frac{\bar{y}_{t'}}{\theta_t}),$$

so the menu $\{(\bar{w}_t, \bar{y}_t)\}_{t=1}^m$ is also downward incentive compatible. By the optimality of $\{(\tilde{w}_t, \tilde{y}_t)\}_{t=1}^m$, it follows that

$$\sum_{i=1}^m p_t E(\tilde{y}_t - \tilde{w}_t) \geq \sum_{t=1}^m p_t(\bar{y}_t - \bar{w}_t)$$

and, since $\bar{y}_t = E\tilde{y}_t$ for all t, that

$$\sum_{i=1}^m p_t E\tilde{w}_t \leq \sum_{t=1}^m p_t \bar{w}_t.$$

By (3.4) and the strict concavity of u, it follows that $(\tilde{w}_t, \tilde{y}_t) = (\bar{w}_t, \bar{y}_t)$ almost surely for all t, so the contract menu $\{(\tilde{w}_t, \tilde{y}_t)\}_{t=1}^m$ is in fact degenerate. ∎

The reasoning in the proof of Lemma 3.2 would not be available if one had to worry about upward as well as downward incentive constraints. In the principal's relaxed problem upward incentive constraints play no role by assumption. To complete the proof of Theorem 3.1 I will show that they also play no role in the principal's original problem. This is the point of the following lemma, which is proved in the Appendix. The lemma rests on the observation that, by Lemma 3.2, the analysis can be limited to deterministic-contract menus in combination with standard properties of the principal's problem with such deterministic-contract menus.

Lemma 3.3 *Assume (A1) and (A2) and let u be strictly concave. Then any solution to the principal's relaxed problem is upward incentive compatible as well as downward incentive compatible and individually rational.*

From Lemma 3.3, one easily derives:

Lemma 3.4 *Assume (A1) and (A2) and let u be strictly concave. Then the principal's relaxed problem and the principal's original problem have the same solutions.*

Proof. Let $\{(\tilde{w}_t, \tilde{y}_t)\}_{t=1}^m$ be a solution to the principal's relaxed problem. Given that the constraint set of the principal's relaxed problem contains the constraint set of his original problem, the value of the objective function (3.1) at $\{(\tilde{w}_t, \tilde{y}_t)\}_{t=1}^m$ is no less than its value at any other contract menu that is feasible and incentive compatible. Given that, by Lemma 3.3, $\{(\tilde{w}_t, \tilde{y}_t)\}_{t=1}^m$ is incentive compatible as well as individually rational, it follows that $\{(\tilde{w}_t, \tilde{y}_t)\}_{t=1}^m$ is a solution to the principal's original problem. Any contract menu $\{(\tilde{w}'_t, \tilde{y}'_t)\}_{t=1}^m$ that also solves the principal's original problem therefore provides the same value of the objective function (3.1) as the solution $\{(\tilde{w}_t, \tilde{y}_t)\}_{t=1}^m$ to the principal's relaxed problem. Because any such alternative solution $\{(\tilde{w}'_t, \tilde{y}'_t)\}_{t=1}^m$ to the principal's original problem is also downward incentive compatible and individually rational, it must also be a solution to the principal's relaxed problem. ∎

Given the identity of the solutions to the principal's relaxed and original problems, Theorem 3.1 is a direct consequence of Lemma 3.2.

Although the principal-agent problem under consideration is quite special, the argument generalizes to all problems with hidden characteristics in which the solutions to the problem as posed are a subset of the solutions to the relaxed problem which is obtained when only downward incentive constraints are considered. In Hellwig (2004 a, 2004 b), I show that this is the case for the optimal income tax problem of Mirrlees (1971, 1976). Under an assumption of weakly decreasing consumption specific risk aversion therefore, it is never desirable to use randomized income tax schemes. The robust conditions for the desirability of randomization that were presented by Stiglitz (1982) and Brito et al. (1995) require the assumption that consumption specific risk aversion is *not* weakly decreasing in the productivity parameter.

For problems in which upward as well as downward incentive constraints are binding, the argument used to prove Lemma 3.2 breaks down so the approach developed here has no bite. Presumably, in such problems, the assertion that randomization is undesirable is not generally valid even if utility functions exhibit monotonic consumption specific risk aversion.

4 Specificity versus Uniformity in Risk Aversion

So far, this paper has introduced the concept of i-premium specific greater risk aversion and shown that this concept is useful for studying the impact of differences in attitudes towards risks when ordinal preferences over the underlying outcome space are not identical. I now turn to the concern of Kihlstrom and Mirman (1974) that, in such situations, the differences in ordinal preferences over the underlying outcome space would seem to vitiate any comparative assessment of risk attitudes. The following example confirms and even sharpens their concern. The example shows that a comparative assessment of risk attitudes on the basis of i-premium specific greater risk aversion is sensitive to the choice of i, i.e. the choice of commodity in which the risk premium is to be paid.

Example 4.1 *Let $n = 2$, and consider the family of utility functions $\{u_\alpha\}$ such that*

$$u_{\alpha\beta}(x_1, x_2) = \alpha \ln x_1 + \beta \ln x_2. \tag{4.1}$$

Then for $\frac{\alpha_1}{\beta_1} < \frac{\alpha_2}{\beta_2}$, the function $u_{\alpha_1\beta_1}$ exhibits 1-premium specific weakly greater risk aversion than the function $u_{\alpha_2\beta_2}$, and the function $u_{\alpha_2\beta_2}$ exhibits 2-premium specific weakly greater risk aversion than the function $u_{\alpha_1\beta_1}$.

Proof. For any $\alpha > 0$ and $\beta > 0$, one has

$$D^2 u_{\alpha\beta}(\mathbf{x}) = \begin{pmatrix} -\alpha\frac{1}{x_1^2} & 0 \\ 0 & -\beta\frac{1}{x_2^2} \end{pmatrix}.$$

Hence,

$$A^1(\mathbf{x}|u_{\alpha\beta}) = \begin{pmatrix} \frac{1}{x_1} & 0 \\ 0 & \frac{\beta}{\alpha}\frac{x_1}{x_2^2} \end{pmatrix} \quad \text{and} \quad A^2(\mathbf{x}|u_{\alpha\beta}) = \begin{pmatrix} \frac{\alpha}{\beta}\frac{x_2}{x_1^2} & 0 \\ 0 & \frac{1}{x_2} \end{pmatrix}.$$

If $\frac{\alpha_1}{\beta_1} < \frac{\alpha_2}{\beta_2}$, one therefore has

$$B^1(\mathbf{x}|u_{\alpha_1\beta_1}, u_{\alpha_2\beta_2}) = \begin{pmatrix} 0 & 0 \\ 0 & \left(\frac{\beta_1}{\alpha_1} - \frac{\beta_2}{\alpha_2}\right)\frac{x_1}{x_2^2} \end{pmatrix} \tag{4.2}$$

and

$$B^2(\mathbf{x}|u_{\alpha_2\beta_2}, u_{\alpha_1\beta_1}) = \begin{pmatrix} (\frac{\alpha_2}{\beta_2} - \frac{\alpha_1}{\beta_1})\frac{x_2}{x_1^2} & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.3}$$

so both, $B^1(\mathbf{x}|u_{\alpha_1\beta_1}, u_{\alpha_2\beta_2})$ and $B^2(\mathbf{x}|u_{\alpha_2\beta_2}, u_{\alpha_1\beta_1})$ are positive semidefinite. By Theorem 2.1, it follows that $u_{\alpha_1\beta_1}$ exhibits 1-premium specific greater risk aversion than $u_{\alpha_2\beta_2}$, and $u_{\alpha_2\beta_2}$ exhibits 2-premium specific greater risk aversion than $u_{\alpha_1\beta_1}$. ■

In this example, curvature considerations are dominated by the differences in ordinal preferences that are induced by differences in the ratio $\frac{\alpha}{\beta}$. The finding that for $\frac{\alpha_1}{\beta_1} < \frac{\alpha_2}{\beta_2}$, the utility function $u_{\alpha_1\beta_1}$ exhibits 1-premium specific weakly greater risk aversion and the utility function $u_{\alpha_2\beta_2}$ exhibits 2-premium specific weakly greater risk aversion seems to have more to do with these differences in ordinal preferences than with risk attitudes, i.e. for $\frac{\alpha_1}{\beta_1} < \frac{\alpha_2}{\beta_2}$, the person with utility function $u_{\alpha_1\beta_1}$ is relatively less concerned about paying a risk premium in units of good 1 - and is therefore willing to pay a higher risk premium - and the person with utility function $u_{\alpha_2\beta_2}$ is relatively less concerned about paying a risk premium in units of good 2.

However, I do not see this finding as calling for a restriction of comparative assessments of risk aversion to situations in which the different von Neumann-Morgenstern utility functions induce the same ordinal preferences on the underlying outcome space. Theorems 2.1 and 3.1 suggest that the concept of i-specific greater risk aversion is useful in the sense of having a clearcut conceptual characterization and lending itself to applications in agency problems with the Single-Crossing Property. In my view therefore, Example 4.1 should be interpreted as a warning to the user of the concept rather than a reason for doing without it. The warning is that in situations involving differences in ordinal preferences over the underlying outcome space, the comparative assessment of risk attitudes through the concept of i-premium specific risk aversion depends on the commodity i in which risk premia are paid. Any analysis based on this concept must therefore make sure that the specification of the commodity for the payment of risk premia is actually appropriate to the situation on hand.

In some contexts, it may not be appropriate to have an *a priori* specification of one commodity for the payment of risk premia. In such contexts, one may want to impose the requirement that the comparison of risk attitudes be independent of how risk premia are paid. For this purpose, I introduce the notion that risk aversion under one utility function is *uniformly* greater than under another. Formally, I say that a utility function u exhibits *uniformly weakly greater risk aversion* than an alternative utility function v, if and only if, for every lottery $\tilde{\mathbf{x}}$ and every vector $\boldsymbol{\pi} \in \Re^n_+$,

$$Ev(\tilde{\mathbf{x}}) = v(E\tilde{\mathbf{x}} - \boldsymbol{\pi}) \text{ implies } Eu(\tilde{\mathbf{x}}) \leq u(E\tilde{\mathbf{x}} - \boldsymbol{\pi}). \tag{4.4}$$

The utility function u exhibits *uniformly strictly greater risk aversion* than v if the inequality in (4.4) is strict for every nondegenerate $\tilde{\mathbf{x}}$ and every i. The following result provides the analogue of Theorem 2.1 for this concept.

Theorem 4.2 *For any two von Neumann-Morgenstern utility functions u and v, the following statements are equivalent:*

(a) u exhibits uniformly weakly greater risk aversion than v;

(b) for $i = 1, ..., n$, there exists a concave function φ^i such that for all $\mathbf{x} \in \Re$,

$$u(\mathbf{x}) = \varphi^i(x_1, ..., x_{i-1}, v(x_1, ..., x_n), x_{i+1}, ..., x_n); \tag{4.5}$$

(c) the matrix

$$B^i(\mathbf{x}|u, v) = -\left[\frac{1}{u_i(\mathbf{x})} D^2 u(\mathbf{x}) - \frac{1}{v_i(\mathbf{x})} D^2 v(\mathbf{x})\right] \tag{4.6}$$

is positive semidefinite for every $\mathbf{x} \in \Re$ and every $i = 1, ..., n$.

Proof. By Theorem 2.1, statements (b) and (c) are equivalent to each other and to the additional statement

$(\hat{a})$ for $i = 1, ..., n$, u exhibits i-premium specific weakly greater risk aversion than v.

Therefore it suffices to prove that $(\hat{a})$ is equivalent to (a). The implication $(a) \Longrightarrow (\hat{a})$ is trivial. To prove the reverse implication, $(\hat{a}) \Longrightarrow (a)$, I note first that $(\hat{a})$ implies the validity of (4.4) for every $\tilde{\mathbf{x}}$ and every $\boldsymbol{\pi} \in \Re^n_+$ that takes the form $\boldsymbol{\pi} = \pi_i \mathbf{e}_i$, where $\mathbf{e}_i$ is the i-th unit vector. In particular, with $\pi_i = \pi_i(\tilde{\mathbf{x}}, v)$, one has

$$Ev(\tilde{\mathbf{x}}) = v(E\tilde{\mathbf{x}} - \pi_i(\tilde{\mathbf{x}}, v)\mathbf{e}_i) \text{ and } Eu(\tilde{\mathbf{x}}) \leq u(E\tilde{\mathbf{x}} - \pi_i(\tilde{\mathbf{x}}, v)\mathbf{e}_i). \tag{4.7}$$

I next show that, for every $\tilde{\mathbf{x}}$ and every $\boldsymbol{\pi} \in \Re^n_+$, under $(\hat{a})$, $Ev(\tilde{\mathbf{x}}) = v(E\tilde{\mathbf{x}} - \boldsymbol{\pi})$ implies $u(E\tilde{\mathbf{x}} - \boldsymbol{\pi}) \geq \min_i u(E\tilde{\mathbf{x}} - \pi_i(\tilde{\mathbf{x}}, v)\mathbf{e}_i)$ and hence, by (4.7), $u(E\tilde{\mathbf{x}} - \boldsymbol{\pi}) \geq Eu(\tilde{\mathbf{x}})$, as required for the validity of (a). Equivalently, I claim that for some i, the vector $\pi_i(\tilde{\mathbf{x}}, v)\mathbf{e}_i$ minimizes $u(E\tilde{\mathbf{x}} - \boldsymbol{\pi})$ over the set of $\boldsymbol{\pi} \in \Re^n_+$ for which $Ev(\tilde{\mathbf{x}}) = v(E\tilde{\mathbf{x}} - \boldsymbol{\pi})$.

To establish this claim, I will show that for any $\boldsymbol{\pi} \in \Re^n_+$ such that $\pi_i > 0$ for more than one index i, there exists some $\hat{\boldsymbol{\pi}} \in \Re^n_+$ such that the set $I^+(\hat{\boldsymbol{\pi}}) := \{i|\pi_i > 0\}$ of indices with strictly positive entries in $\hat{\boldsymbol{\pi}}$ is a strict subset of the set $I^+(\boldsymbol{\pi}) := \{i|\pi_i > 0\}$ of indices with strictly positive entries in $\boldsymbol{\pi}$ and, moreover, $v(E\tilde{\mathbf{x}} - \hat{\boldsymbol{\pi}}) = v(E\tilde{\mathbf{x}} - \boldsymbol{\pi})$ and $u(E\tilde{\mathbf{x}} - \hat{\boldsymbol{\pi}}) \leq u(E\tilde{\mathbf{x}} - \boldsymbol{\pi})$. In other words, u is not increased if one moves from $\boldsymbol{\pi}$ to a suitably chosen point $\hat{\boldsymbol{\pi}}$ which yields the same v and which has at least one more zero than $\boldsymbol{\pi}$. Upon repeating the operation, if necessary, one also finds that u is not increased if one moves from $\boldsymbol{\pi}$ to a suitably chosen point $\check{\boldsymbol{\pi}}$ which yields the same v and which has no more than one nonzero entry, i.e. which takes the form $\check{\boldsymbol{\pi}} = \pi_i \mathbf{e}_i$ for some i.

To prove that, under $(\hat{a})$, u is not increased if one moves from $\boldsymbol{\pi}$ to a suitably chosen point $\hat{\boldsymbol{\pi}}$ which yields the same v and which has at least one more zero than $\boldsymbol{\pi}$, fix $\boldsymbol{\pi} \in \Re^n_+$, and consider the map $\hat{\pi}_{j_1} \rightarrow \boldsymbol{\pi}(\hat{\pi}_{j_1})$ which is defined by setting

$$\pi_i(\hat{\pi}_{j_1}) = \pi_i \quad \text{for all } i \neq j_1, j_2, \tag{4.8}$$

$$\pi_{j_1}(\hat{\pi}_{j_1}) = \hat{\pi}_{j_1}, \tag{4.9}$$

and

$$v(E\tilde{\mathbf{x}} - \boldsymbol{\pi}(\hat{\pi}_{j_1})) = v(E\tilde{\mathbf{x}} - \boldsymbol{\pi}), \tag{4.10}$$

the component $\pi_{j_2}(\hat{\pi}_{j_1})$ of the vector $\boldsymbol{\pi}(\hat{\pi}_{j_1})$ being implicitly defined by (4.10). By the implicit function theorem, one has

$$\frac{d\pi_{j_2}(\hat{\pi}_{j_1})}{d\hat{\pi}_{j_1}} = -\frac{v_{j_1}(E\tilde{\mathbf{x}} - \boldsymbol{\pi}(\hat{\pi}_{j_1}))}{v_{j_2}(E\tilde{\mathbf{x}} - \boldsymbol{\pi}(\hat{\pi}_{j_1}))} \tag{4.11}$$

and

$$\frac{d^2\pi_{j_2}(\hat{\pi}_{j_1})}{d\hat{\pi}_{j_1}^2} = \frac{1}{v_{j_2}^2}\left[v_{j_2}v_{j_1j_1} - v_{j_1}v_{j_2j_1} + (v_{j_2}v_{j_1j_2} - v_{j_1}v_{j_2j_2})\frac{d\pi_{j_2}}{d\hat{\pi}_{j_1}}\right], \tag{4.12}$$

which is conveniently rewritten in the form

$$\frac{d^2\pi_{j_2}}{d\hat{\pi}_{j_1}^2} = \frac{1}{v_{j_2}}\left[v_{j_1j_1} + (v_{j_2j_1} + v_{j_1j_2})\frac{d\pi_{j_2}}{d\hat{\pi}_{j_1}} + v_{j_2j_2}\left[\frac{d\pi_{j_2}}{d\hat{\pi}_{j_1}}\right]^2\right]. \tag{4.13}$$

For the function $\hat{\pi}_{j_1} \rightarrow w(\hat{\pi}_{j_1}) := u(E\tilde{\mathbf{x}} - \boldsymbol{\pi}(\hat{\pi}_{j_1}))$, one obtains

$$\frac{dw}{d\hat{\pi}_{j_1}} = -u_{j_1} - u_{j_2}\frac{d\pi_{j_2}}{d\hat{\pi}_{j_1}} \tag{4.14}$$

and

$$\frac{d^2 w}{d\hat{\pi}_{j_1}^2} = u_{j_1 j_1} + u_{j_2 j_1}\frac{d\pi_{j_2}}{d\hat{\pi}_{j_1}} + u_{j_1 j_2}\frac{d\pi_{j_2}}{d\hat{\pi}_{j_1}} + u_{j_2 j_2}\left[\frac{d\pi_{j_2}}{d\hat{\pi}_{j_1}}\right]^2 - u_{j_2}\frac{d^2\pi_{j_2}}{d\hat{\pi}_{j_1}^2}. \tag{4.15}$$

From (4.15) and (4.13), one obtains

$$\frac{1}{u_{j_2}}\frac{d^2 w}{d\hat{\pi}_{j_1}^2} = +\frac{1}{u_{j_2}}\left[u_{j_1 j_1} + (u_{j_2 j_1} + u_{j_1 j_2})\frac{d\pi_{j_2}}{d\hat{\pi}_{j_1}} + u_{j_2 j_2}\left[\frac{d\pi_{j_2}}{d\hat{\pi}_{j_1}}\right]^2\right] - \frac{1}{v_{j_2}}\left[v_{j_1 j_1} + (v_{j_2 j_1} + v_{j_1 j_2})\frac{d\pi_{j_2}}{d\hat{\pi}_{j_1}} + v_{j_2 j_2}\left[\frac{d\pi_{j_2}}{d\hat{\pi}_{j_1}}\right]^2\right]. \tag{4.16}$$

Given the equivalence of statements $(\hat{a})$ and (c), under $(\hat{a})$, the difference of quadratic forms on the right-hand side of (4.16) is nonpositive. The function $\hat{\pi}_{j_1} \to w(\hat{\pi}_{j_1}) := u(E\tilde{\mathbf{x}} - \boldsymbol{\pi}(\hat{\pi}_{j_1}))$ is therefore concave and attains a minimum on the boundary, at $\hat{\pi}_{j_1}^*$ satisfying $\hat{\pi}_{j_1}^* = 0$ or $\pi_{j_2}(\hat{\pi}_{j_1}^*) = 0$. The point $\hat{\boldsymbol{\pi}} = \boldsymbol{\pi}(\hat{\pi}_{j_1}^*)$ then has one more zero entry than the original point $\boldsymbol{\pi}$ and, by construction, one has $v(E\tilde{\mathbf{x}} - \hat{\boldsymbol{\pi}}) = v(E\tilde{\mathbf{x}} - \boldsymbol{\pi})$ and $u(E\tilde{\mathbf{x}} - \hat{\boldsymbol{\pi}}) \leq u(E\tilde{\mathbf{x}} - \boldsymbol{\pi})$.
■

At this point, the reader may wonder how the concept of uniformly greater risk aversion relates to the analysis of Kihlstrom and Mirman (1974). According to Kihlstrom and Mirman, a utility function u exhibits weakly greater (strictly greater) risk aversion than a utility function v if one can write $u = \varphi \circ v$, where φ is a concave (strictly concave) function. Any two functions u and v that are comparable by the Kihlstrom-Mirman criterion thus satisfy statement (b) in Theorem 4.2 with $\varphi^i = \varphi$, independent of i. By Theorem 4.2, it follows that if u exhibits weakly greater risk aversion than v by the Kihlstrom-Mirman criterion, then u must also exhibit uniformly weakly greater risk aversion than v by the criterion defined here.

However, the converse is not true. There are instances of utility functions u and v such that u exhibits uniformly greater risk aversion than v and yet, u and v are not comparable by the Kihlstrom-Mirman criterion. Comparability of risk aversion in the sense of Theorem 4.2 does *not* require the von Neumann-Morgenstern utility functions to induce the same preference ordering on the underlying outcome space. In the following example, utility functions are comparable in the sense that one exhibits uniformly greater risk aversion than the other and, yet, they are not ordinally equivalent. For the comparison of risk attitudes to be independent of how risk premia are paid one does not have to impose ordinal equivalence.

Example 4.3 *Let* $n = 2$, *fix* $\alpha > 0$, $\beta > 0$, *and consider the family of utility functions* $\{u_{\gamma\delta}\}$ *such that*

$$u_{\gamma\delta}(x_1, x_2) = \alpha \ln x_1 + \beta \ln x_2 + \gamma x_1 + \delta x_2. \tag{4.17}$$

Then for $\gamma_1 < \gamma_2$ *and* $\delta_1 < \delta_2$, *the utility function* $u_{\gamma_1\delta_1}$ *exhibits uniformly strictly greater risk aversion than the utility function* $u_{\gamma_2\delta_2}$.

Proof. For any γ and δ, one has

$$Du_{\gamma\delta}(\mathbf{x}) = \begin{pmatrix} \frac{\alpha}{x_1} + \gamma \\ \frac{\beta}{x_2} + \delta \end{pmatrix} \tag{4.18}$$

and

$$D^2 u_{\gamma\delta}(\mathbf{x}) = \begin{pmatrix} -\alpha\frac{1}{x_1^2} & 0 \\ 0 & -\beta\frac{1}{x_2^2} \end{pmatrix}. \tag{4.19}$$

Therefore,

$$A^1(\mathbf{x}|u_{\gamma\delta}) = \frac{x_1}{\alpha + \gamma x_1} \begin{pmatrix} \alpha\frac{1}{x_1^2} & 0 \\ 0 & \beta\frac{1}{x_2^2} \end{pmatrix} \tag{4.20}$$

and

$$A^2(\mathbf{x}|u_{\alpha\beta}) = \frac{x_2}{\beta + \delta x_2} \begin{pmatrix} \alpha\frac{1}{x_1^2} & 0 \\ 0 & \beta\frac{1}{x_2^2} \end{pmatrix}. \tag{4.21}$$

It follows that

$$B^1(\mathbf{x}|u_{\gamma_1\delta_1}, u_{\gamma_2\delta_2}) = \frac{(\gamma_2 - \gamma_1)x_1^2}{(\alpha + \gamma_1 x_1)(\alpha + \gamma_2 x_1)} \begin{pmatrix} \alpha\frac{1}{x_1^2} & 0 \\ 0 & \beta\frac{1}{x_2^2} \end{pmatrix} \tag{4.22}$$

and

$$B^2(\mathbf{x}|u_{\gamma_1\delta_1}, u_{\gamma_2\delta_2}) = \frac{(\delta_2 - \delta_1)x_2^2}{(\beta + \delta_1 x_2)(\beta + \delta_2 x_2)} \begin{pmatrix} \alpha\frac{1}{x_1^2} & 0 \\ 0 & \beta\frac{1}{x_2^2} \end{pmatrix}, \tag{4.23}$$

so, if $\gamma_1 < \gamma_2$ and $\delta_1 < \delta_2$, both, $B^1(\mathbf{x}|u_{\gamma_1\delta_1}, u_{\gamma_2\delta_2})$ and $B^2(\mathbf{x}|u_{\gamma_1\delta_1}, u_{\gamma_2\delta_2})$ are positive definite. By a straightforward adaptation of the reasoning yielding the implication $(c) \Rightarrow (a)$ in Theorem 4.2, this implies that $u_{\gamma_1\delta_1}$ exhibits uniformly strictly greater risk aversion than $u_{\gamma_2\delta_2}$. ■

A Appendix: Proof of Lemma 3.3

By Lemma 3.2, any solution $\{(w_t, y_t)\}_{t=1}^m$ to the principal's relaxed problem is also a solution to the relaxed deterministic-contracts problem of choosing $\{(w_t, y_t)\}_{t=1}^m$ to maximize

$$\sum_{t=1}^{m} p_t(y_t - w_t) \tag{A.1}$$

subject to

$$u(w_t, 1 - \frac{y_t}{\theta_t}) \geq \bar{u} \tag{A.2}$$

for all t and

$$u(w_t, 1 - \frac{y_t}{\theta_t}) \geq u(w_{t'}, 1 - \frac{y_{t'}}{\theta_t}) \tag{A.3}$$

for all t and all $t' < t$. To prove Lemma 3.3, it is therefore sufficient to show that any solution to the relaxed deterministic-contracts problem is incentive compatible, i.e. satisfies (A.3) for all t and all t' rather than just all $t' < t$. The argument proceeds in several steps. The overall proof strategy follows Matthews and Moore (1987) or Hellwig (2004a).

Lemma A.1 *Assume (A1) and (A2) and let u be strictly concave. Then any solution $\{(w_t, y_t)\}_{t=1}^m$ to the principal's relaxed problem satisfies*

$$u_1(w_t, 1 - \frac{y_t}{\theta_t}) - u_2(w_t, 1 - \frac{y_t}{\theta_t})\frac{1}{\theta_t} \geq 0 \tag{A.4}$$

for $t = 1, .., m$.

Proof. Let $\{(w_t, y_t)\}_{t=1}^m$ be any solution to the principal's relaxed problem. For any t let $u_t^* := u(w_t, 1 - \frac{y_t}{\theta_t})$ and, relying on the strict concavity of u, set

$$(w_t^*, y_t^*) := \arg \max_{u(w, 1 - \frac{y}{\theta_t}) = u_t^*} (y - w) \tag{A.5}$$

and

$$\hat{w}_t := \min(w_t^*, w_t), \hat{y}_t := \min(y_t^*, y_t). \tag{A.6}$$

Under the given monotonicity assumptions, one has $w_t > w_t^*$ if and only if $y_t > y_t^*$, hence $\hat{w}_t = w_t$ if and only if $\hat{y}_t > y_t$. It follows that

$$u(\hat{w}_t, 1 - \frac{\hat{y}_t}{\theta_t}) = u(w_t, 1 - \frac{y_t}{\theta_t}) \tag{A.7}$$

for all t. Trivially then, for any t, the validity of (A.2) implies that $u(\hat{w}_t, 1 - \frac{\hat{y}_t}{\theta_t}) \geq \bar{u}$. The deterministic-contracts menu $\{(\hat{w}_t, \hat{y}_t)\}_{t=1}^m$ is individually rational.

Given that $\hat{y}_t \geq y_t$, (A.7) in combination with the Single-Crossing Property (A1) also implies that

$$u(\hat{w}_t, 1 - \frac{\hat{y}_t}{\theta_{\hat{t}}}) \leq u(w_t, 1 - \frac{y_t}{\theta_{\hat{t}}}) \tag{A.8}$$

for all $\hat{t} > t$. For any t and any $t' < t$, the validity of (A.3) and (A.7) therefore imply

$$u(\hat{w}_t, 1 - \frac{\hat{y}_t}{\theta_t}) = u(w_t, 1 - \frac{y_t}{\theta_t}) \geq u(w_{t'}, 1 - \frac{y_{t'}}{\theta_t}) \geq u(\hat{w}_{t'}, 1 - \frac{\hat{y}_{t'}}{\theta_t}),$$

so the menu $\{(\hat{w}_t, \hat{y}_t)\}_{t=1}^m$ is also downward incentive compatible.

By the optimality of $\{(w_t, y_t)\}_{t=1}^m$ over the set of individually rational and downward incentive compatible deterministic-contracts menus, it follows that

$$\sum_{i=1}^m p_t(y_t - w_t) \geq \sum_{t=1}^m p_t(\hat{y}_t - \hat{w}_t). \tag{A.9}$$

Because the construction of $\{(\hat{w}_t, \hat{y}_t)\}_{t=1}^m$ implies $\hat{y}_t - \hat{w}_t \geq y_t - w_t$ for all t, it follows that $\hat{y}_t - \hat{w}_t = y_t - w_t$ for all t. Hence $(w_t, y_t) \leq (w_t^*, y_t^*)$ for all t. (A.4) then follows from the first-order conditions for (w_t^*, y_t^*). ■

Lemma A.2 *Assume (A1) and (A2) and let u be strictly concave. Then any solution $\{(w_t, y_t)\}_{t=1}^m$ to the principal's relaxed problem satisfies*

$$u(w_t, 1 - \frac{y_t}{\theta_t}) = u(w_{t-1}, 1 - \frac{y_{t-1}}{\theta_t}) \tag{A.10}$$

and

$$y_t \geq y_{t-1} \tag{A.11}$$

for $t = 2, ..., m$.

Proof. The proof proceeds by induction on t. For $t = 1$, there is nothing to prove. For $t > 1$, suppose that (A.10) and (A.11) have been verified for $t' < t$ and consider the choice of the pair (w_t, y_t). By the induction hypothesis, one has $y_1 \leq ...y_{t-1}$, and by downward incentive compatibility,

$$u(w_{t-1}, 1 - \frac{y_{t-1}}{\theta_{t-1}}) \geq u(w_{t'}, 1 - \frac{y_{t'}}{\theta_{t-1}}) \tag{A.12}$$

for all $t' < t-1$. By the Single-Crossing Property (A1), it follows that

$$u(w_{t-1}, 1 - \frac{y_{t-1}}{\theta_t}) \geq u(w_{t'}, 1 - \frac{y_{t'}}{\theta_t}) \tag{A.13}$$

for all $t' < t-1$, so the downward incentive compatibility condition (A.3) is satisfied for t and $t' < t$ if it is satisfied for t and $t-1$. Similarly, by the monotonicity of u and the validity of (A.2) for $t-1$, the individual rationality condition (A.2) for t is satisfied if downward incentive compatibility is satisfied for t and $t-1$.

Apart from the downward incentive constraints linking $\hat{t} > t$ and t, the downward incentive constraint for t and $t-1$ is thus the only constraint to be considered in choosing (w_t, y_t). Given that a decrease in w_t or an increase in y_t raises the value of the principal's objective while leaving unaffected the validity of the downward incentive constraints linking $\hat{t} > t$ and t, the downward incentive constraint for t and $t-1$ provides the only consideration putting a stop to such a decrease in w_t or an increase in y_t. At the optimal (w_t, y_t), this constraint must therefore be binding. Thus (A.10) must hold for t as well as $t' < t$.

As for the inequality (A.11), I note that, by Lemma A.1 applied to (w_{t-1}, y_{t-1}) and by the Single-Crossing Property (A1), one has $y - w < y_{t-1} - w_{t-1}$ for all (w, y) such that $y < y_{t-1}$ and

$$u(w, 1 - \frac{y}{\theta_t}) = u(w_{t-1}, 1 - \frac{y_{t-1}}{\theta_t}).$$

Thus $(w_t, y_t) < (w_{t-1}, y_{t-1})$ and (A.10) would imply that the principal can raise his profits by replacing (w_t, y_t) for type t by (w_{t-1}, y_{t-1}). Since this is inconsistent with the optimality of the menu $\{(w_t, y_t)\}_{t=1}^m$, it follows that $w_t \geq w_{t-1}$ and $y_t \geq y_{t-1}$, i.e. (A.11) holds for t as well as $t' < t$. ■

Lemma A.3 *Assume (A1) and (A2) and let u be strictly concave. Then any solution $\{(w_t, y_t)\}_{t=1}^m$ to the principal's relaxed problem is upward incentive compatible.*

Proof. By the Single-Crossing Property (A1), for any t, (A.10) implies

$$u(w_t, 1 - \frac{y_t}{\theta_{t-1}}) \leq u(w_{t-1}, 1 - \frac{y_{t-1}}{\theta_{t-1}}). \tag{A.14}$$

For any t and any $t' > t$, one thus has

$$u(w_\tau, 1 - \frac{y_\tau}{\theta_{\tau-1}}) \leq u(w_{\tau-1}, 1 - \frac{y_{\tau-1}}{\theta_{\tau-1}}) \tag{A.15}$$

and, by (A.11), $y_\tau \geq y_{\tau-1}$ for $\tau = t+1, ..., t'$. By another application of the Single-Crossing Property (A1), it follows that, for any t and any $t' > t$, one has

$$u(w_\tau, 1 - \frac{y_\tau}{\theta_t}) \leq u(w_{\tau-1}, 1 - \frac{y_{\tau-1}}{\theta_t}) \tag{A.16}$$

for $\tau = t+1, ..., t'$. Upon combining these inequalities for $\tau = t+1, ..., t'$, one obtains

$$u(w_{t'}, 1 - \frac{y_{t'}}{\theta_t}) \leq u(w_t, 1 - \frac{y_t}{\theta_t}), \tag{A.17}$$

which is just the condition for upward incentive compatibility. ■

References

[1] Arrow, K.J. (1965), *Aspects of the Theory of Risk-Bearing*, Yrjö Jahnsson Foundation, Helsinki.

[2] Brito, D. L., J.H. Hamilton, S.M. Slutsky, and J.E. Stiglitz (1995), Randomization in Optimal Income Tax Schedules, *Journal of Public Economics* 56, 189 - 223.

[3] Edlin, A.S., and C.M. Shannon (1998), Strict Single Crossing and the Strict Spence-Mirrlees Condition: A Comment on Monotone Comparative Statics, *Econometrica* 66, 1417 - 1425.

[4] Fudenberg, D., and J. Tirole (1991), *Game Theory*, MIT Press, Cambridge.

[5] Hellwig, M.F. (2004a), A Contribution to the Theory of Optimal Income Taxation, mimeo, Max Planck Institute for Research on Collective Goods, Bonn.

[6] Hellwig, M.F. (2004b), Nonincreasing Risk Aversion and the Undesirability of Randomization in Optimal Income Taxation, mimeo, Max Planck Institute for Research on Collective Goods, Bonn.

[7] Kihlstrom, R.E., and L.J. Mirman (1974), Risk Aversion with Many Commodities, *Journal of Economic Theory* 8, 361 - 388.

[8] Kihlstrom, R.E., and L.J. Mirman (1981), Constant, Increasing, Decreasing Risk Aversion with Many Commodities, *Review of Economic Studies* 48, 271 - 280.

[9] Maskin, E., and J. Riley (1984), Optimal Auctions with Risk Averse Buyers, *Econometrica* 52, 1473 - 1518.

[10] Matthews, S., and J. Moore (1987), Monopoly Provision of Quality and Warranties: An Exploration in the Theory of Multidimensional Screening, *Econometrica* 55, 441 - 467.

[11] Milgrom, P.R., and C.M. Shannon (1994), Monotone Comparative Statics, *Econometrica* 62, 157 - 180.

[12] Mirrlees, J.M. (1971), An Exploration in the Theory of Optimum Income Taxation, *Review of Economic Studies* 38, 175 - 208.

[13] Mirrlees, J.M. (1976), Optimal Tax Theory: A Synthesis, *Journal of Public Economics* 6, 327 - 358.

[14] Pratt, J.W. (1964), Risk Aversion in the Small and in the Large, *Econometrica* 32, 122 - 136.

[15] Stiglitz, J.E. (1969), Behavior Towards Risk with Many Commodities, *Econometrica* 37, 660 - 667.

[16] Stiglitz, J.E. (1982), Self-Selection and Pareto Efficient Taxation, *Journal of Public Economics* 17, 213 - 240.